AM-FM Analysis of Medical Images

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Talk Outline

- An Introduction
- Frequency Modulation
- An AM-FM Model for M-Mode Ultrasound
- Fast AM-FM Demodulation Implementation
- Discrete-Space Orthogonal FM Transforms
  - application to the multidimensional DFT
- Concluding Remarks

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The image and video Processing and Communications Lab (*ivPCL*) founded 2000

Ph.D. students (3 advanced to candidacy):

- S. Cai: "Tumor Growth Measurement in Lung Cancer Imaging"
- J. Kern: "Multispectral Image Registration using Mutual Information"
- P. Rodriguez V.: "High-Performance Signal and Image Processing"

M.S. students (6):

- J. Ramachandran: "Hierarchical Lung Image Segmentation"
- H. Muralidharan: "Medical Imaging Models with Applications to the Lung and the Heart"
- S. Barriga: "Signal Detection in Retinal Video"
- H. Yu: "3-D Ultrasound"
- A. Vera: "Real-time Video Processing using FPGAs"
- Grant Martin: "3-D CT and Chest Radiograph Image Analysis"

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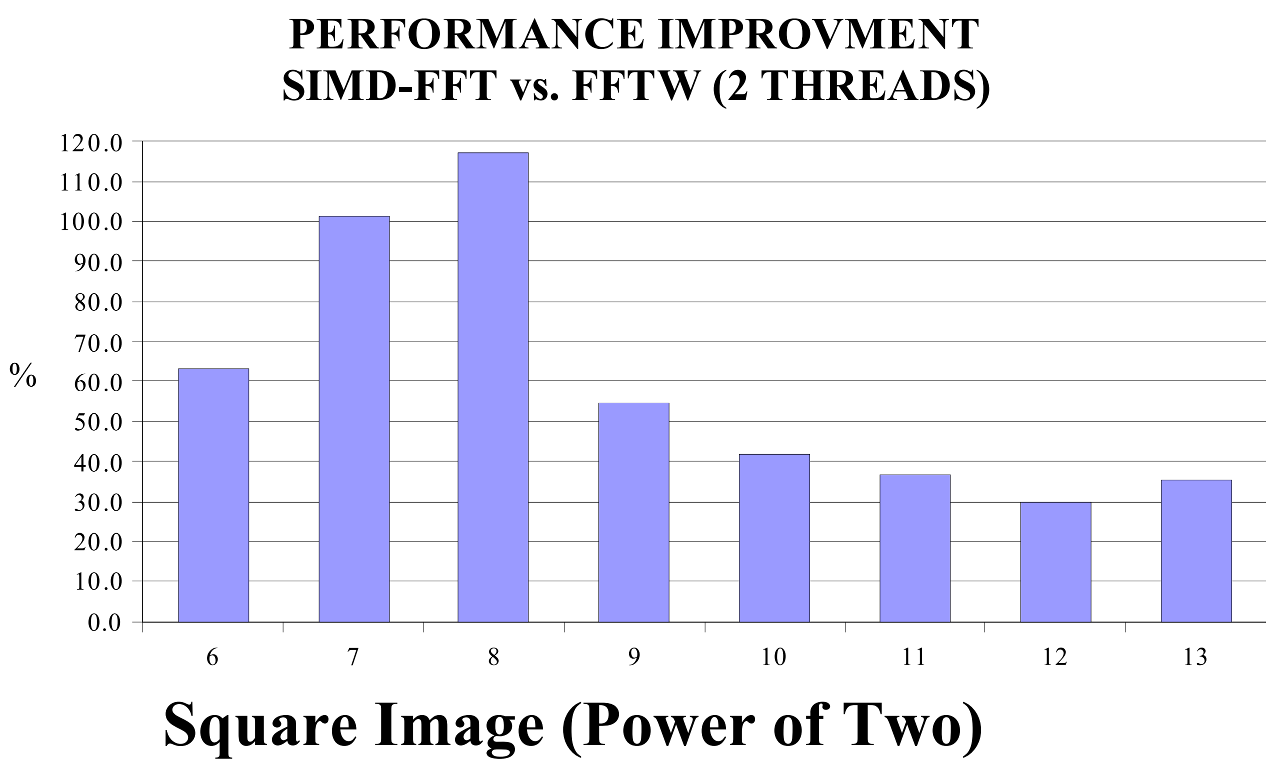
Paul Rodriguez V.'s Poster



SIMD is part of ALL GENERAL MICROPROCESSOR ARCHITECTURES (Pentium III/4 and PowerPC tested)

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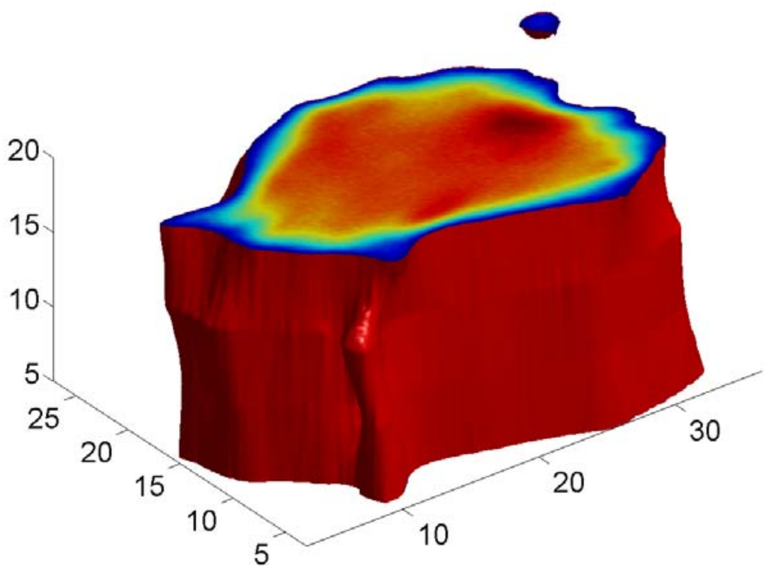
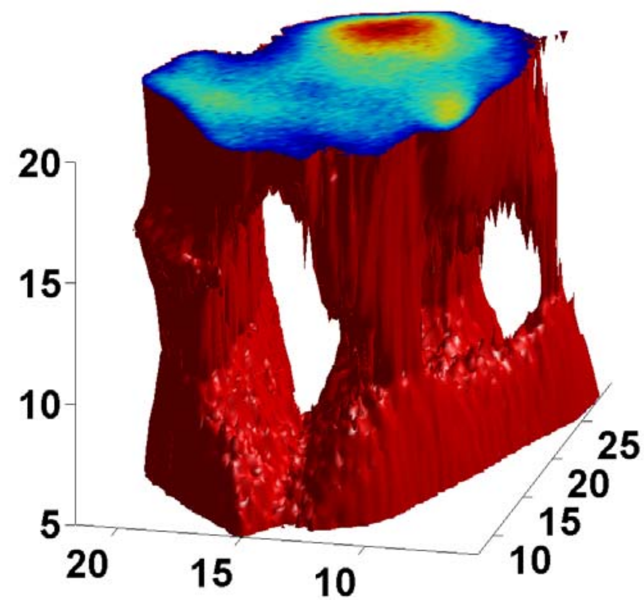
Songhe Cai's Poster
3-D Tumor Reconstruction from CT



Threshold at level=100.

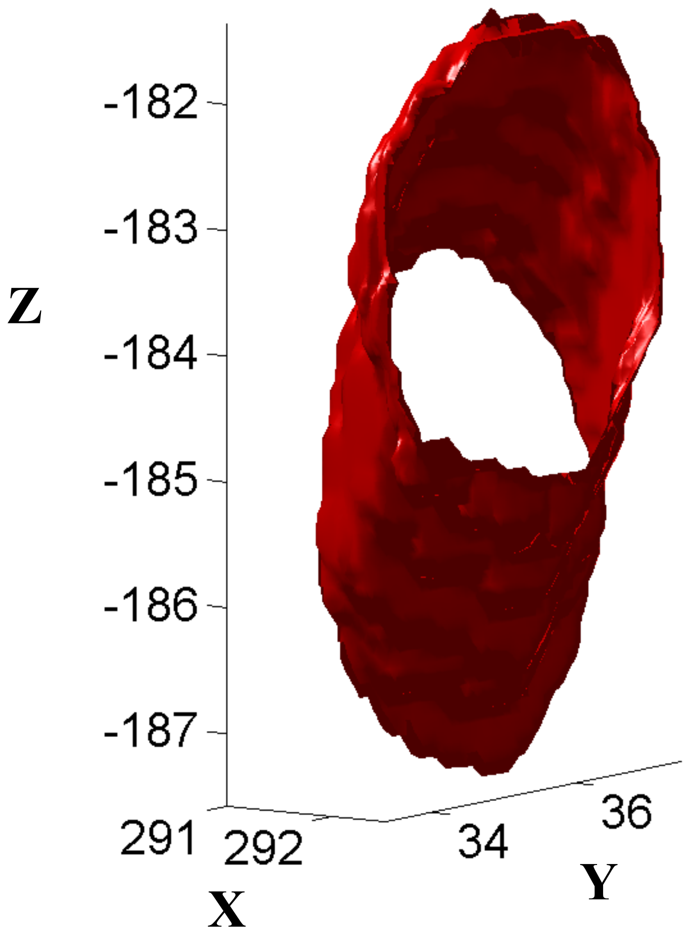


Threshold at level=170.

Early lung tumor growth detection.

Honggang Yu's Poster

3-D Reconstruction from Freehand Ultrasound



Close agreement to cylinder ground truth.

Example demonstrates ability to reconstruct 3-D objects accurately, from arbitrary slices (freehand).

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Image Processing in the EECE Department

Masters and Ph.D. programs in Electrical and Computer Engineering:

- Image Processing track, some related courses:
  - EECE 533 Digital Image Processing
  - EECE 595.1 Medical Imaging
  - EECE 595.2 Advanced Topics in Image Processing
- Computational Intelligence track, some related courses:
  - EECE 517 Pattern Recognition
  - EECE 547 Neural Networks
- Some other related courses:
  - EECE 433 Computer Graphics
  - EECE 516 Computer Vision
  - EECE 595.3 Detection and Estimation Theory

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An Introduction to AM-FM

Related Research

Originated by Teager and ***Kaiser*** for Speech signal analysis

Continued by three main groups:

- ***Bovik*** and his students (***Havlicek, Pattichis***, Sanghoon, …)
  - very low bitrate video coding (MPEG-4, H.263)
  - error resilient and perceptually optimal
  - texture completion using Reaction-Diffusion PDEs
  - fingerprint classification, latent print analysis (Pattichis)
  - shape from texture, …
- ***Maragos and his students*** (Potamianos, ***Sabathanam, …***)
  - 1-D, Speech signal analysis
- Quatieri (1-D, Speech signal analysis)
- Many joined! (1-D, 2-D)

Also, in 1-D, time-frequency analysis and Communications theory.

Continuous-Space AM-FM Transforms

Assume:

- the Fourier transform of image $g(.)$ exists.
- a coordinate transformation $\mathbf{\Phi}(.)$:
  $(x_1, x_2) \longrightarrow \mathbf{\Phi}(x_1, x_2) \equiv (\phi_1(x_1, x_2), \phi_2(x_1, x_2))$,
- an amplitude function $a(.) > 0$.

Then, $g(.)$ can be expressed as:

$$g(\mathbf{x}) = \iint \mathbf{G}_{\mathbf{\Phi}}(\mathbf{f})\, a(\mathbf{x}) e^{j2\pi \mathbf{f}^T \mathbf{\Phi}(\mathbf{x})}\, d\mathbf{f}$$

where $\mathbf{G}_{\mathbf{\Phi}}(.)$ denotes the AM-FM spectrum:

$$\mathbf{G}_{\mathbf{\Phi}}(\mathbf{f}) = \iint \frac{g(\mathbf{x})}{a(\mathbf{x})} e^{-j2\pi \mathbf{f}^T \mathbf{\Phi}(\mathbf{x})}\, |J_{\mathbf{\Phi}}(\mathbf{x})|\, d\mathbf{x}$$

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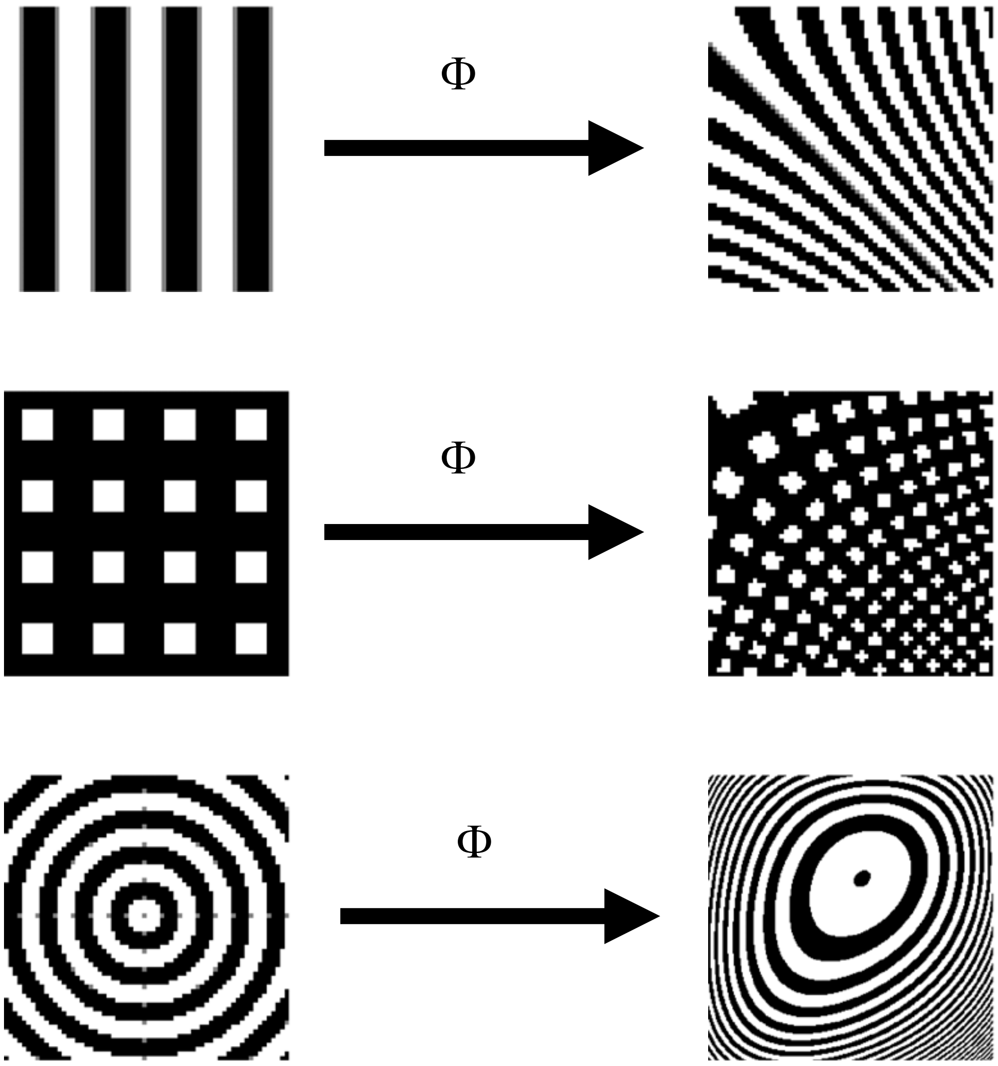
Coordinate Transformation Model



A Summary of AM-FM Demodulation

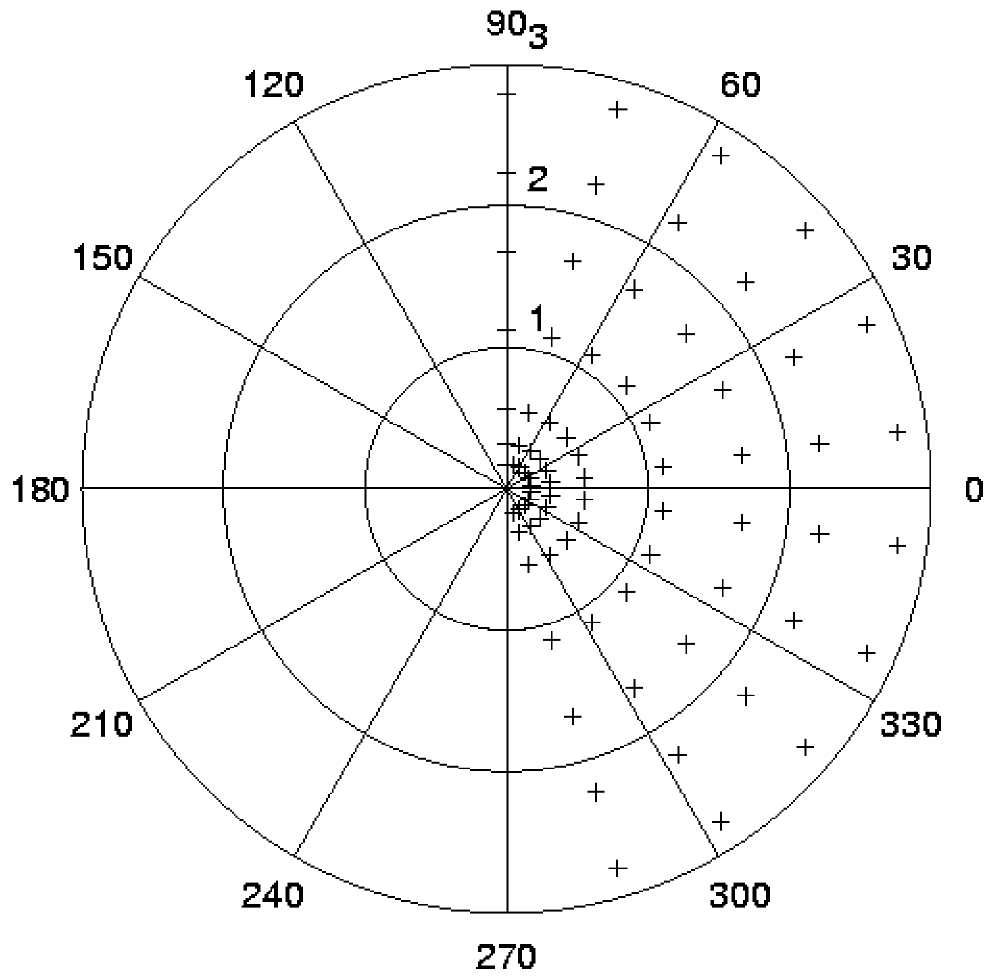
Step 1. Apply bandpass channel filters

Step 2. Compute AM-FM parameters over each channel

Step 3. Reconstruct AM-FM image using channels that produce maximum amplitude estimate (pixel adaptive)

For Teager-Kaiser approach, use maximum energy estimate to select bands.

Gabor filter centers



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AM-FM Demodulation Using Teager-Kaiser Energy Operators

Define the 2-D Teager operator by

$$\Psi_c\{I\}(\mathrm{x}) = \| \nabla I(\mathrm{x}) \| - I(\mathrm{x})\nabla^2 I(\mathrm{x})$$

(where ∇^2 denotes the Laplacian operator).

From $I(x_1, x_2) = A\cos\Theta(x_1, x_2)$, demodulate using:

$$\begin{aligned}
\partial\Theta/\partial x_1 &\approx \sqrt{\Psi_c\{\partial I/\partial x_1\} \;/\; \Psi_c\{I\}}, \\
\partial\Theta/\partial x_2 &\approx \sqrt{\Psi_c\{\partial I/\partial x_2\} \;/\; \Psi_c\{I\}}, \\
|A| &\approx \Psi_c\{I\}/\sqrt{\Psi_c\{\partial I/\partial x_1\} + \Psi_c\{\partial I/\partial x_2\}}
\end{aligned}$$

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AM-FM Demodulation Using Hilbert Approach

For the Hilbert demodulation approach, we use the algorithms developed by Joebob Havlicek in his dissertation.

Using the output from the channel $G_m(\cdot)$ with the maximum energy $t_m(\cdot)$, we estimate:

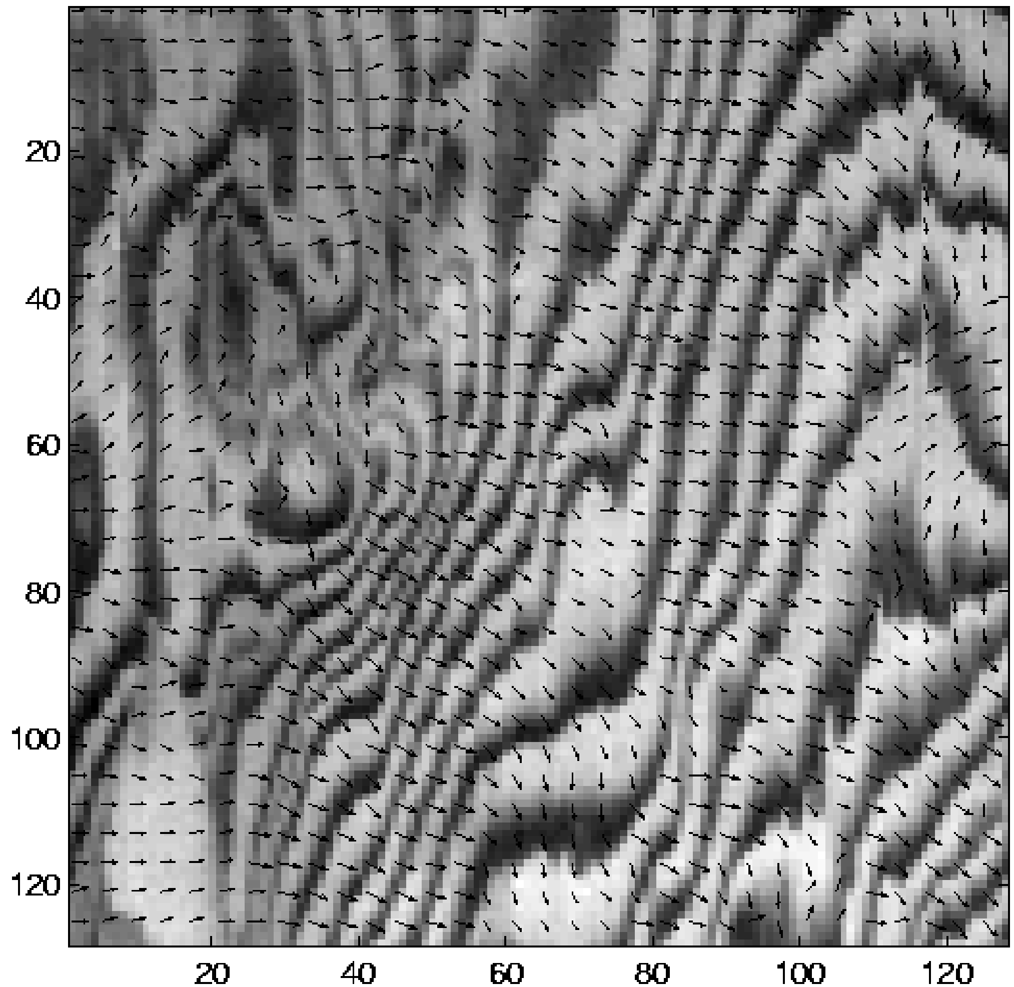
$$\nabla\Theta(\mathrm{x}) \approx \mathrm{Re}\left[\frac{\nabla t_m(\mathrm{x})}{j t_m(\mathrm{x})}\right]$$

$$\Theta(\mathrm{x}) \approx \arctan\left\{\frac{\mathrm{Im}[t_m(\mathrm{x})]}{\mathrm{Re}[t_m(\mathrm{x})]}\right\}$$

$$a(\mathrm{x}) \approx \left|\frac{t_m(\mathrm{x})}{G_m[\nabla\widehat{\theta}(\mathrm{x})]}\right|$$

Basic Examples

Woodgrain image example

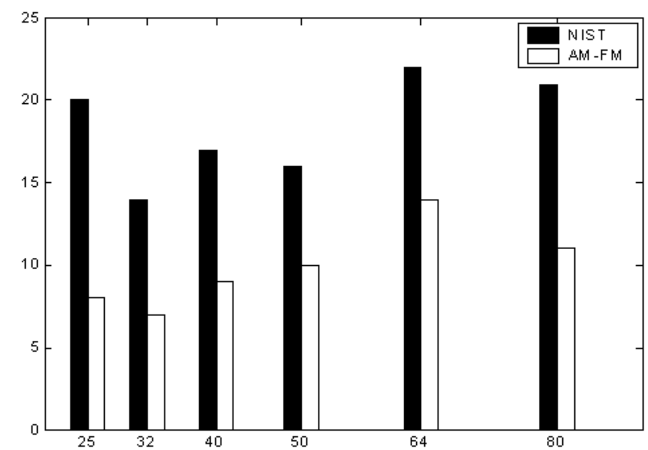


Instantaneous frequency vectors shown (log-scaled and subsampled).

Fingerprint Classification

NIST

FM



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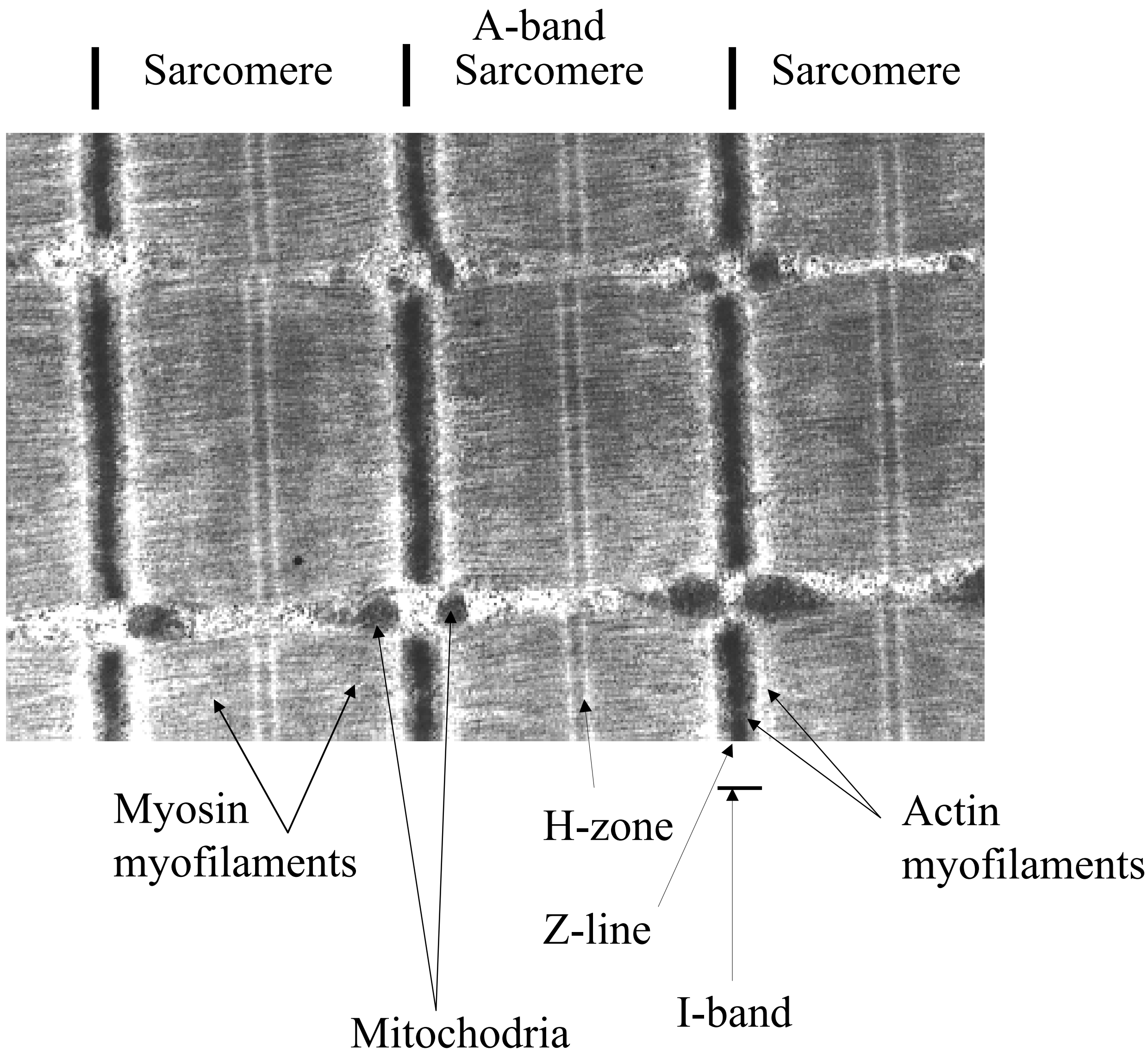
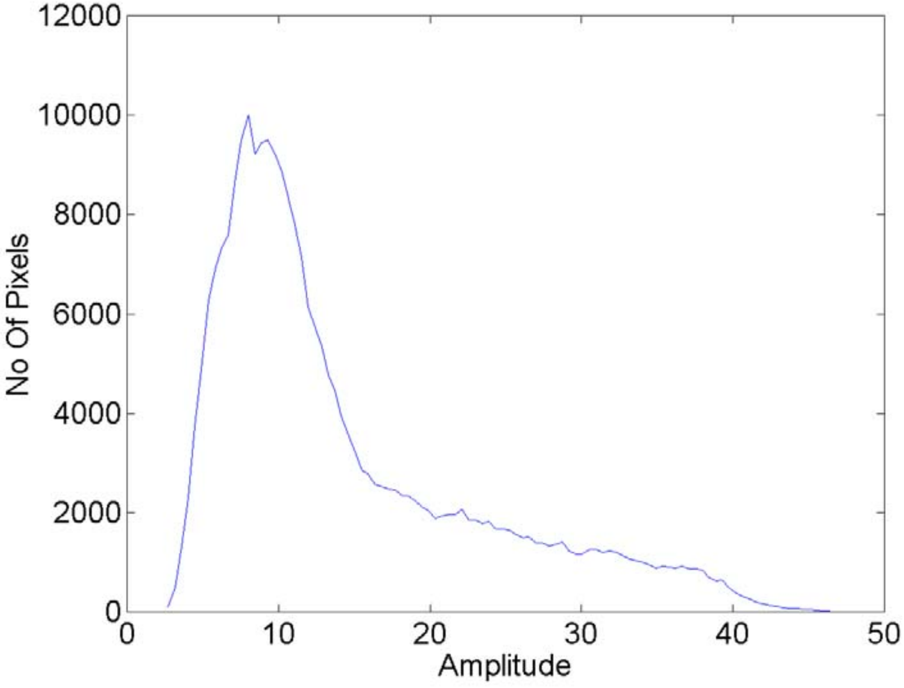
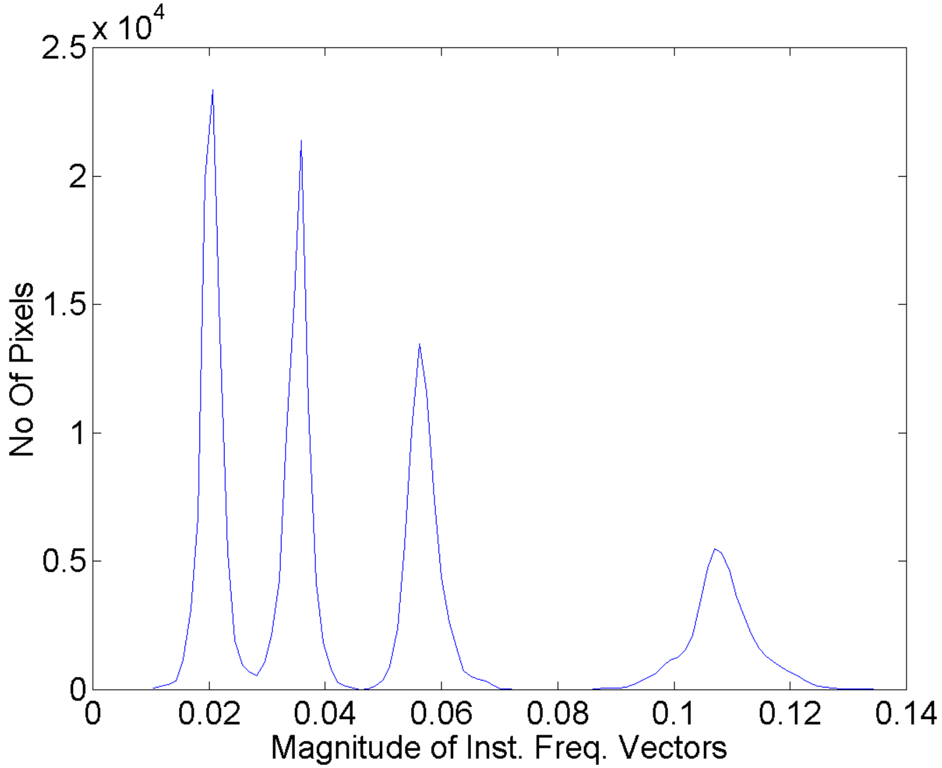
Electron micrograph of human skeletal muscle



Histogram distributions of AM-FM parameters





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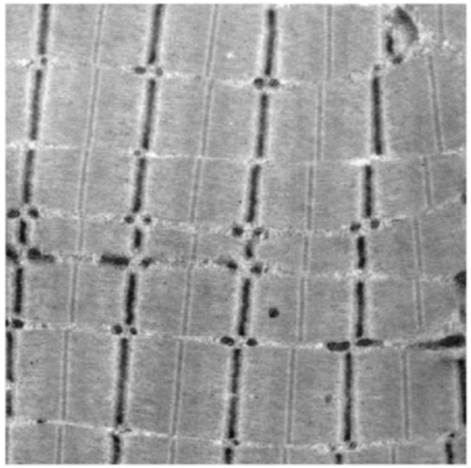
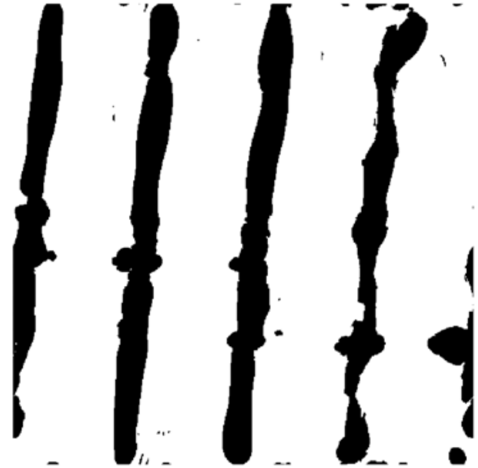
Segmentation



Original Image

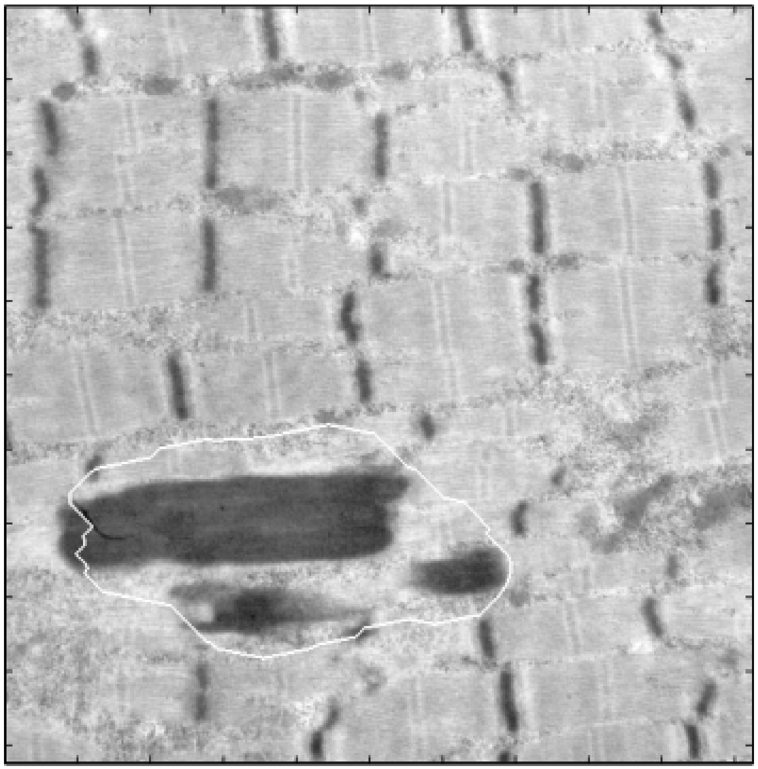


$a(x,y)<20$

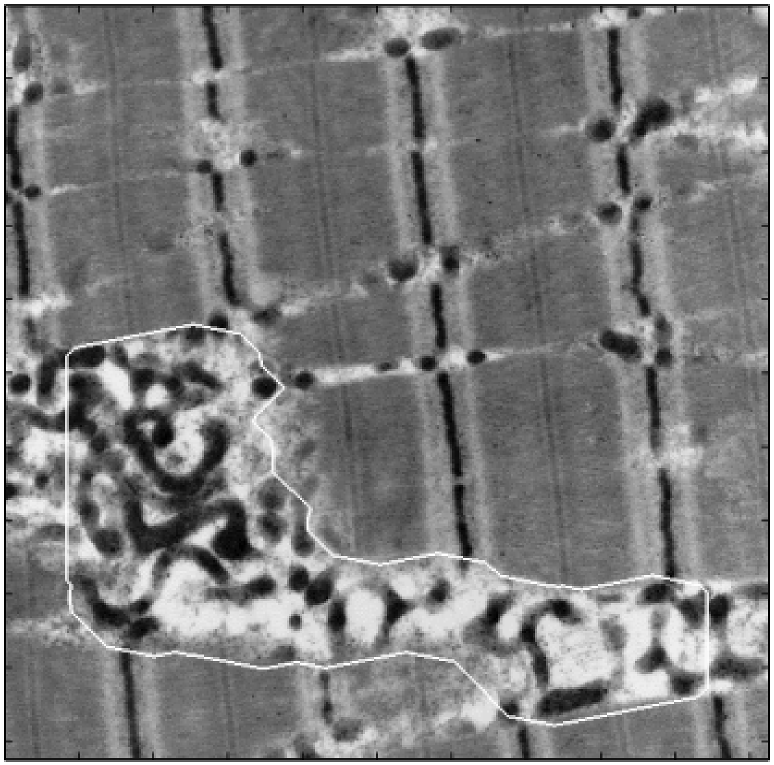
$$\left\|\nabla\phi(x,y)\right\| \le 0.046$$

$$0.08 \le \left\|\nabla\phi(x,y)\right\| \le 0.15$$

Bayesian Segmentation and ASF Filtering using AM-FM Parameters



Nemaline myopathy



Mitochondrial myopathy

Recognition Accuracy

	Type of myopathy		
	Nemaline	Tubular Aggregates	Mitochondrial
Number of cases	2	1	4
Number of regions	10	6	10
Recognition accuracy	84%	78%	75%

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Multidimensional Frequency Modulation

The Instantaneous Frequency Gradient Tensor

Let the instantaneous frequency be given by:

$$\mathbf{O} \equiv \nabla \Theta = (\, O_1,\, O_2 \,)$$

The spatial differential of $\mathbf{O}$ is:

$$\begin{aligned} d\mathbf{O} &= \mathbf{F}\, d\mathbf{x} \\ \mathbf{F} &= \begin{bmatrix} \partial O_1/\partial x_1 & \partial O_1/\partial x_2 \\ \partial O_2/\partial x_1 & \partial O_2/\partial x_2 \end{bmatrix} \end{aligned}$$

is the <u>Instantaneous Frequency Gradient Tensor</u>.

Phase Modulation

Expand the phase in a local Taylor series expansion:

$$\begin{aligned}\Theta(\mathbf{x}) &= \Theta(\mathbf{x}_0) + (\mathbf{x}-\mathbf{x}_0)^T\ \nabla\Theta(\mathbf{x}_0) \\ &+\frac{1}{2}(\mathbf{x}-\mathbf{x}_0)^T\ \mathbf{F}(\mathbf{x}_0)\ (\mathbf{x}-\mathbf{x}_0) \\ &+R_2(\mathbf{x}-\mathbf{x}_0, \mathbf{x}_0),\end{aligned}$$

where: $R_2(\mathbf{x}-\mathbf{x}_0, \mathbf{x}_0) \to 0$ as $\mathbf{x}-\mathbf{x}_0 \to \mathbf{0}$.

Phase modulation is <u>maximized</u> in the direction of $\nabla\Theta$.

Eigen Decomposition of the IFGT

Frequency Modulation expressed in terms of the eigen decomposition of $\mathbf{F}$ (which is real-symmetric):

$$\begin{bmatrix} dO_1 \\ dO_2 \end{bmatrix} = \begin{bmatrix} e_{1,1} & e_{2,1} \\ e_{1,2} & e_{2,2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

$$d\mathbf{O} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^T d\mathbf{x}$$

Frequency Modulation Bounds

From

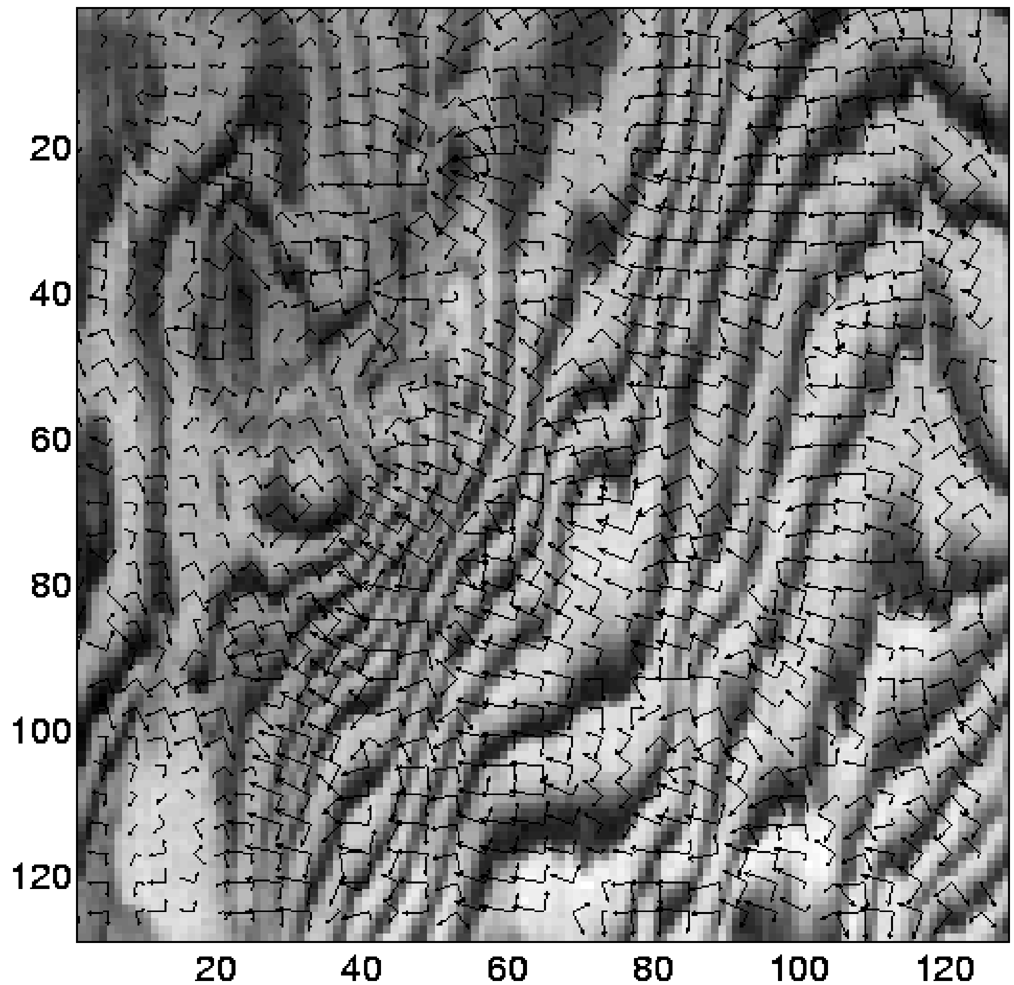
$$d\mathbf{O} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^T d\mathbf{x},$$

we get

$$|\lambda_1| \geq \frac{|d\,\mathbf{O}|}{|d\mathbf{x}|} \geq |\lambda_2|, \qquad |\lambda_1| \geq |\lambda_2|$$

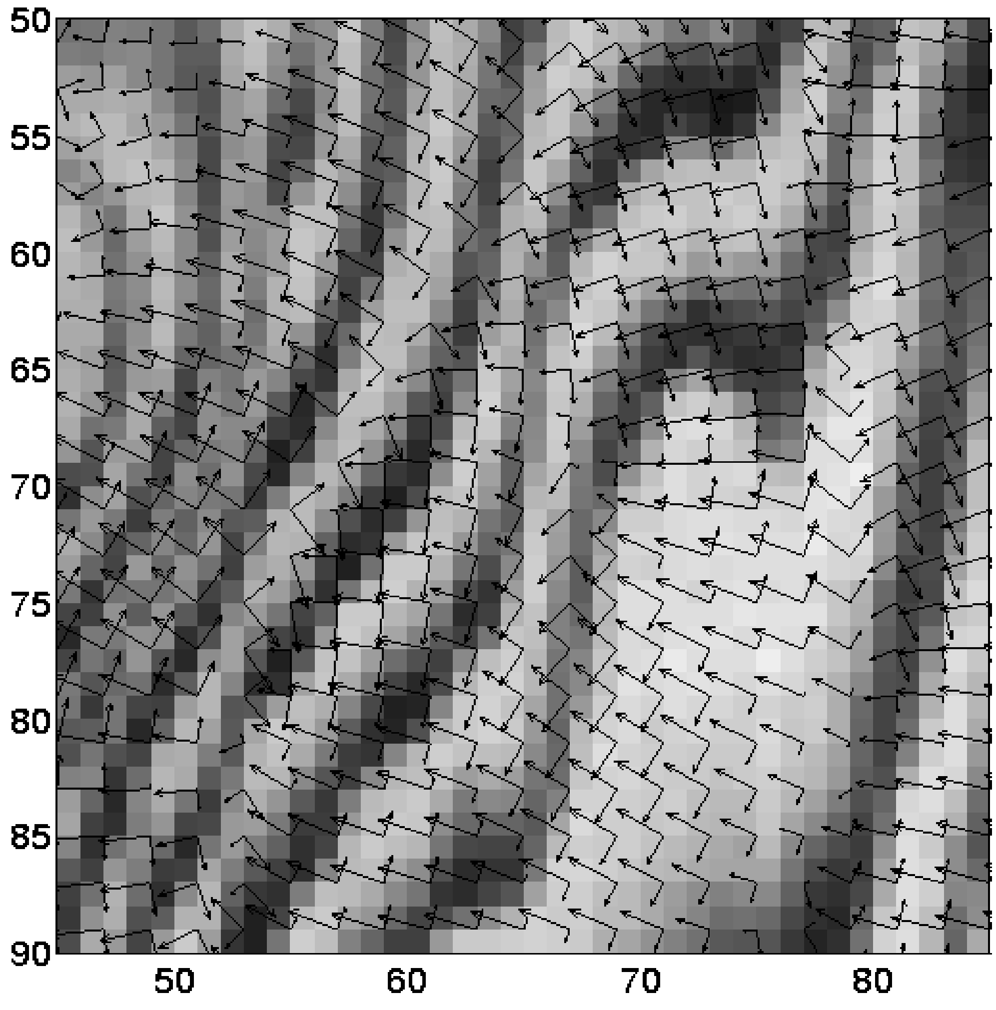
Greatest rate of change of the instantaneous frequency magnitude in direction of eigenvector corresponding to maximum absolute eigenvalue.

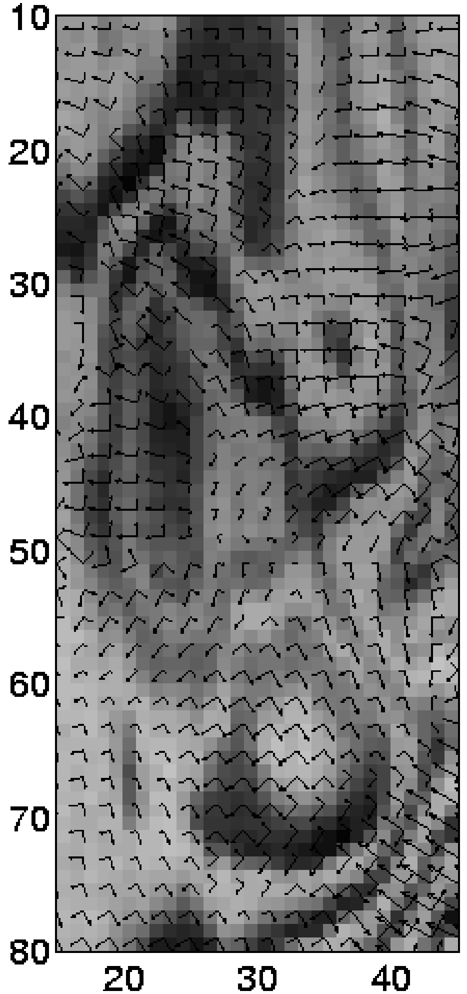
Woodgrain image results



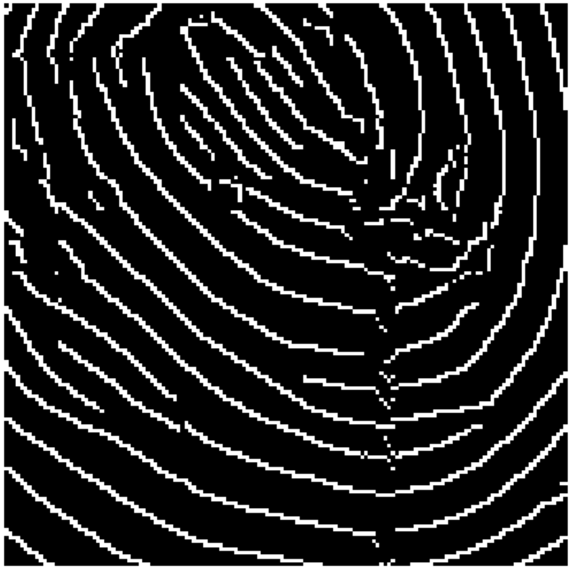
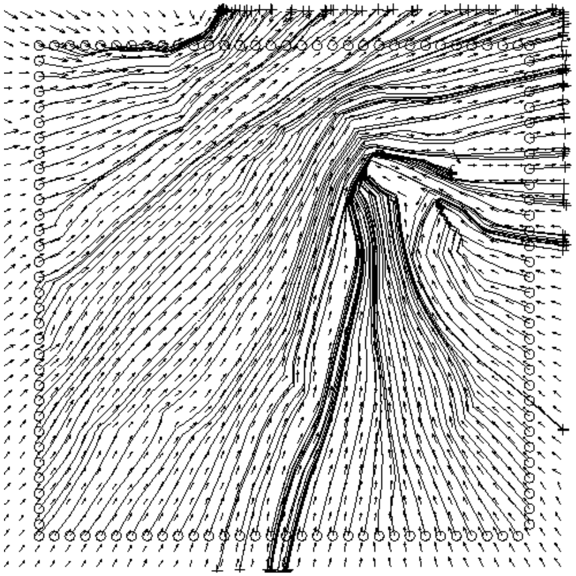
Subsampled $\lambda_1 e_1$, $\lambda_2 e_2$ vectors shown.

Woodgrain image results (Contd)

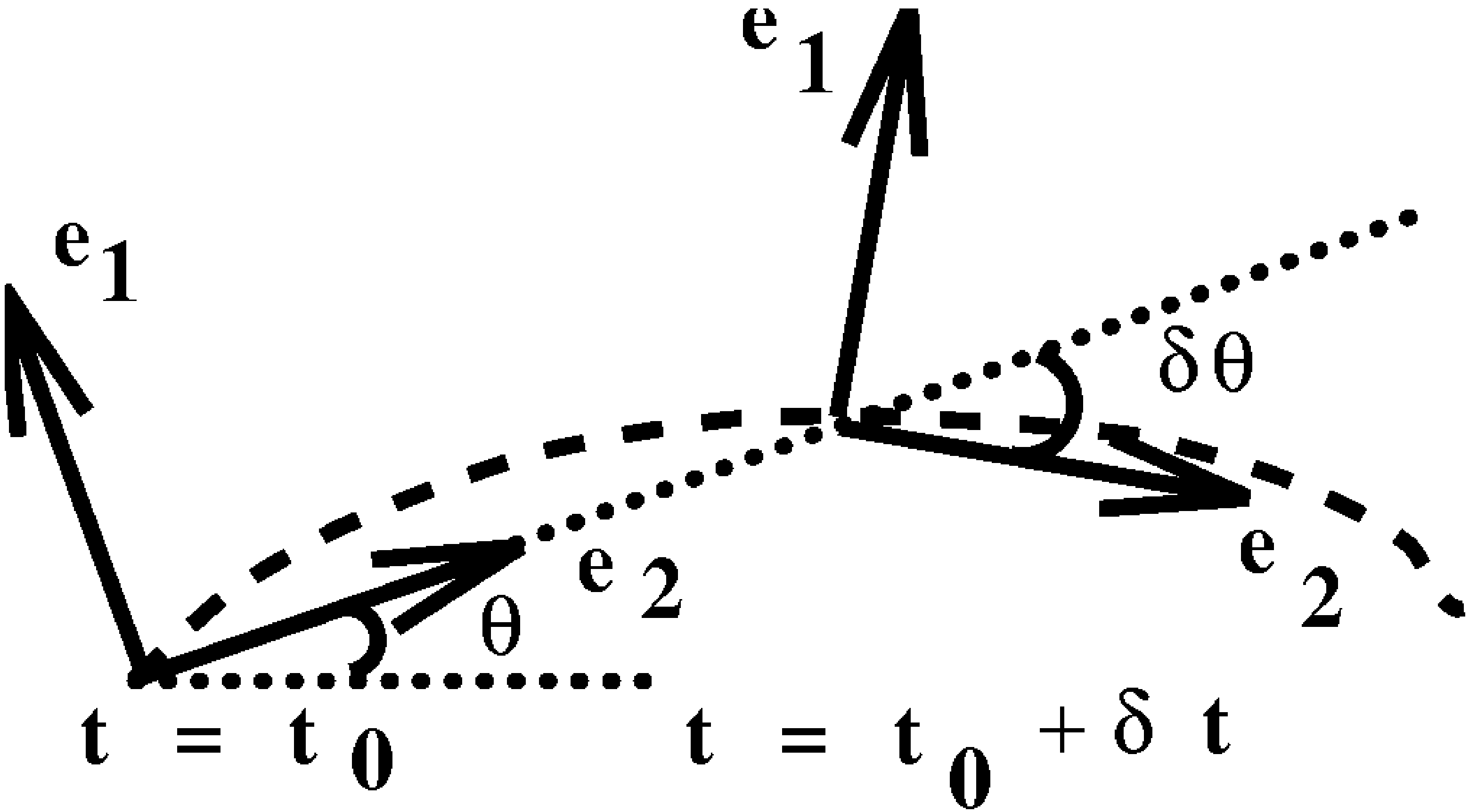




A fingerprint example

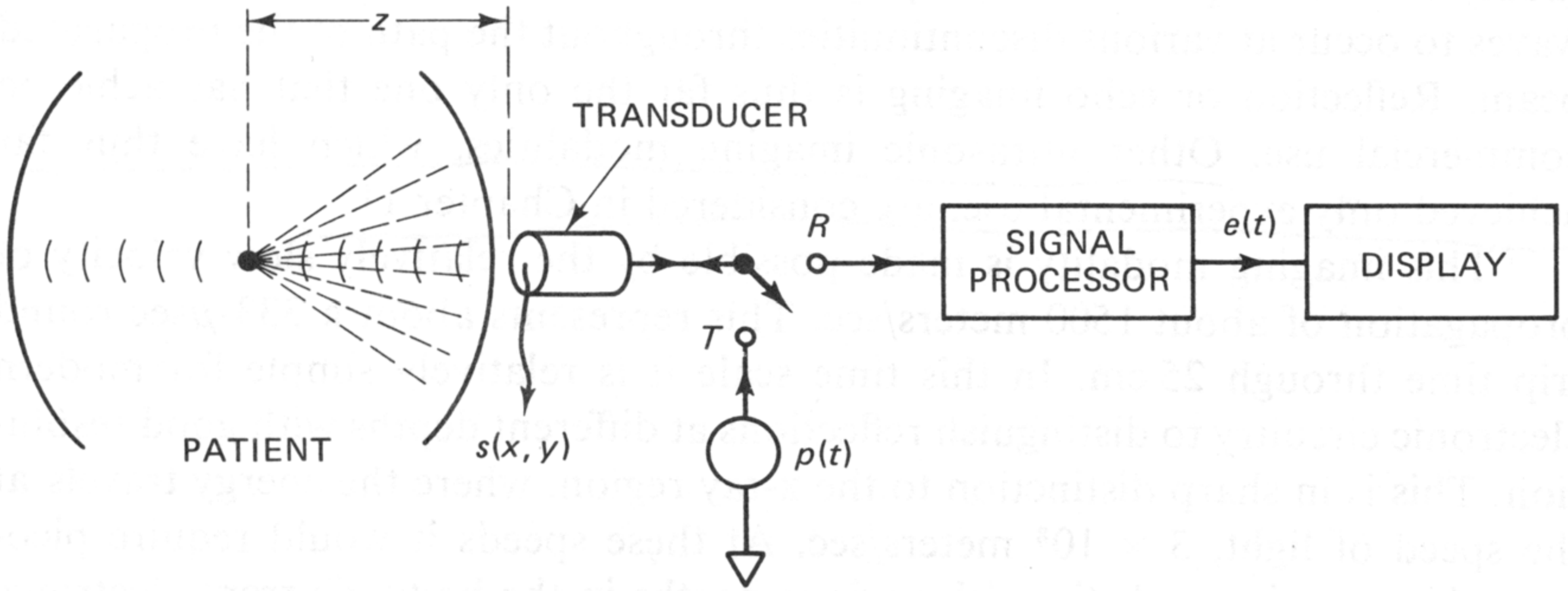
  

Separability Assumption: Slowly-rotating eigenvectors



An AM-FM Model for M-Mode Ultrasound Images

Basic Reflection Imaging System



Transducer acts as both a transmitter and receiver of acoustic waves.

Figure taken from Page 174 of "Medical Imaging Systems" by Albert Macovski.

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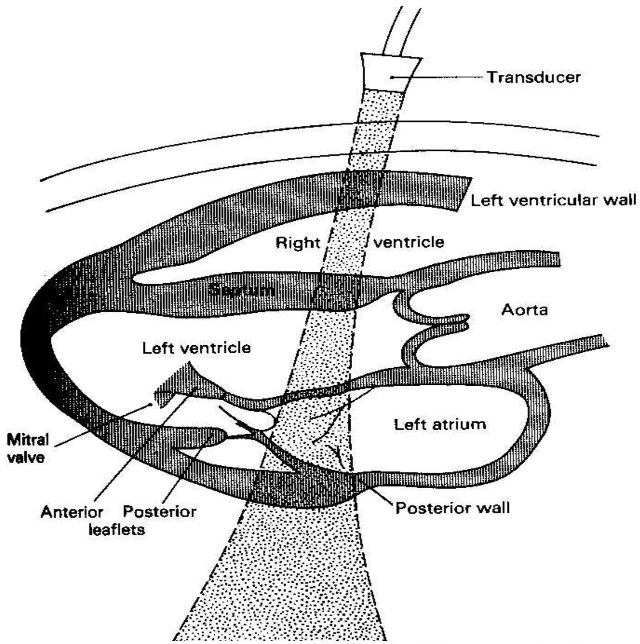
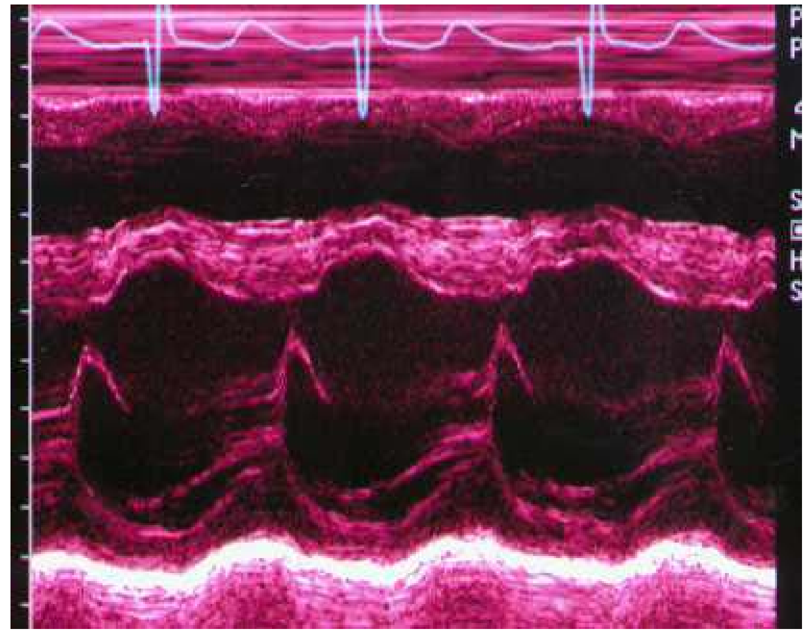
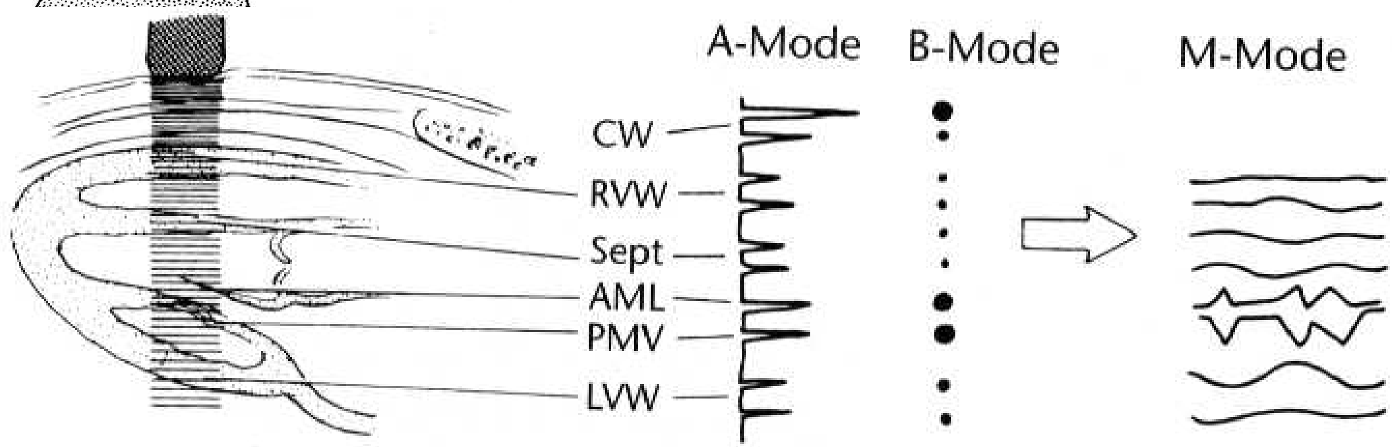
M-mode

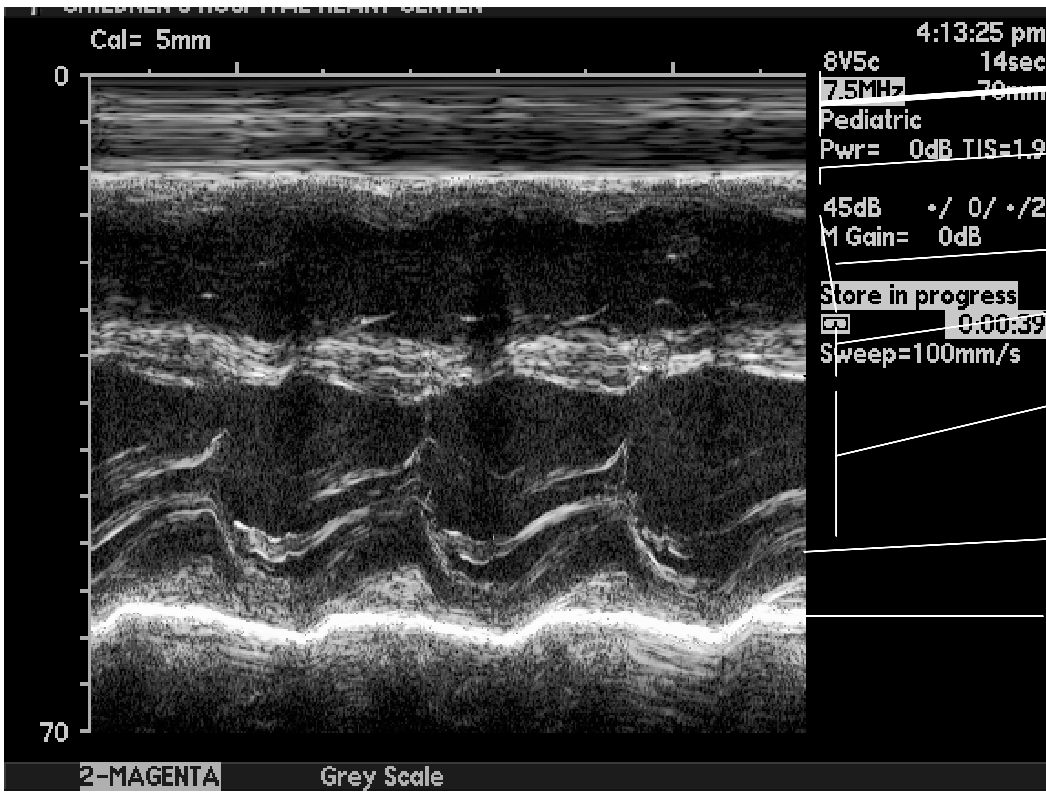






Slide provided by M.B. Goens, M.D.

Ground Truth over an M-mode image



- Near field
- Anterior RV wall
- RV chamber
- Interventricular septum
- LV chamber with some mitral valve apparatus
- LV posterior wall
- Bright epicardium

Image annotated by M.B. Goens, M.D.

An AM-FM Model I

We model M-mode images using an AM-FM series

$$f(y,t)=\sum_{n} C_n a(y,t)\cos[n\phi(y,t)]$$

where $t=x$, $a(y,t)$ denotes the amplitude function, $\phi(y,t)$ denotes the phase function, and C_n denotes the AM-FM series coefficients.

An AM-FM Model II

$$a(y,t)$$

will track the brightness variation due to “material change”, reflected in changes in the reflectivity of the material.

$$\phi(y,t)$$

will track the curvilinear coordinate represented by wall boundaries.

$$a(y,t)\cos[n\phi(y,t)]$$

will track the AM-FM harmonics over the curvilinear coordinate system represented by the wall boundaries.

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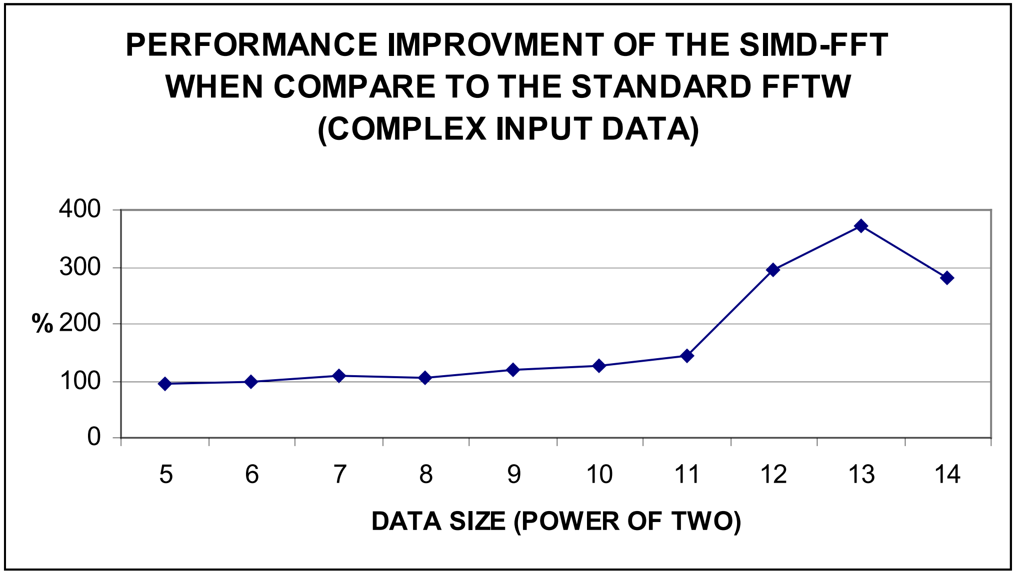
Fast AM-FM Demodulation Implementation

SIMD-FFT (Paul Rodriguez V.)

Comparison between the FFTW (scalar implementation) and SIMD-FFT

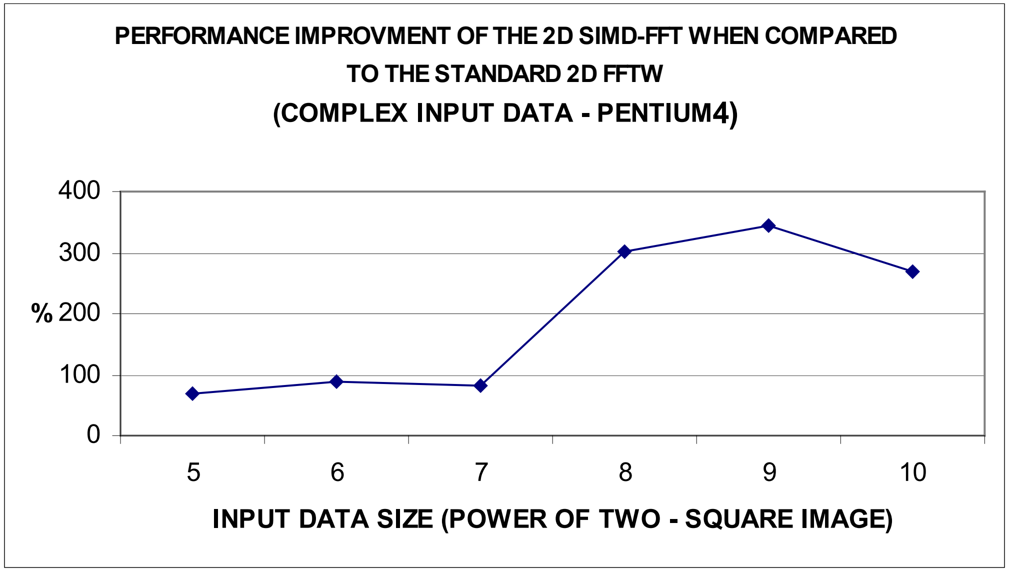
Complex input data – Linux (Intel Architecture)
Improvement range 89% - 374%



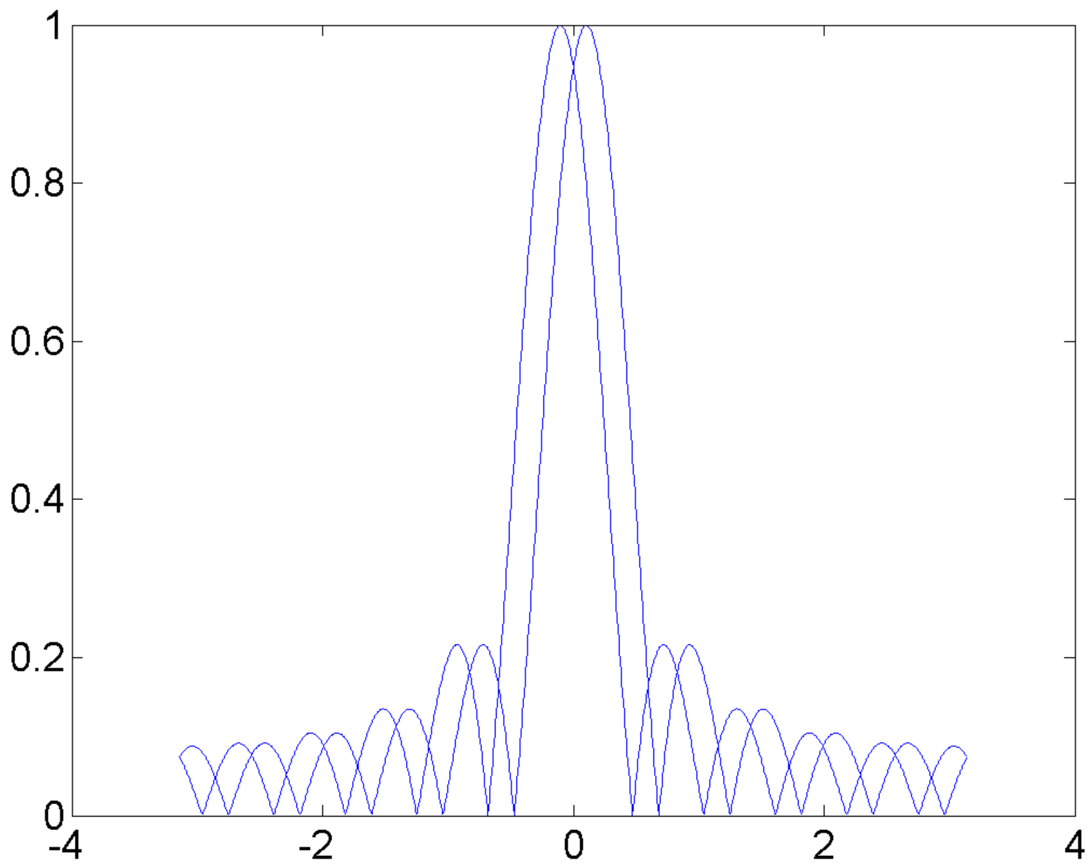
2D SIMD-FFT (Paul Rodriguez V.)

Comparison between the 2D FFTW (scalar implementation) and 2D SIMD-FFT

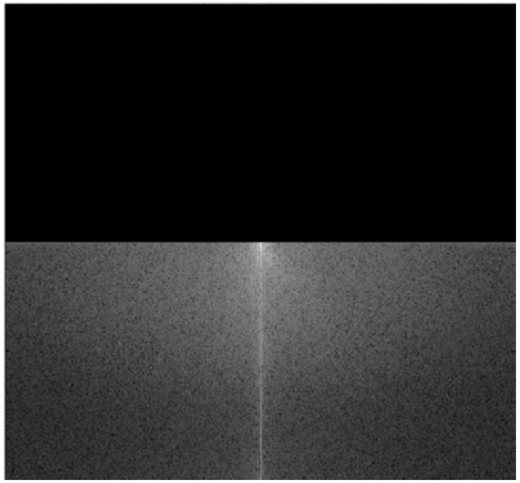
Complex input data – Linux (Intel Architecture)
Improvement range 87% - 330%



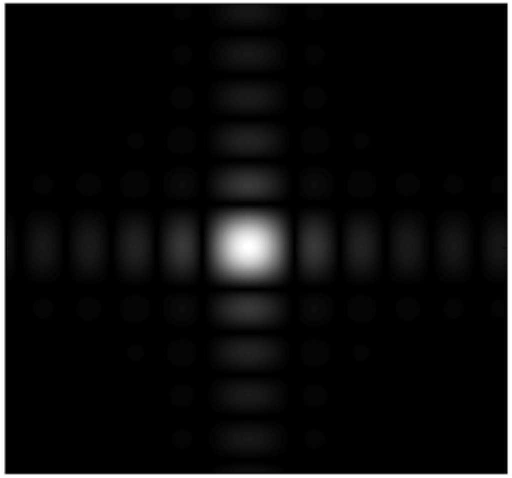
Separable Filter Design



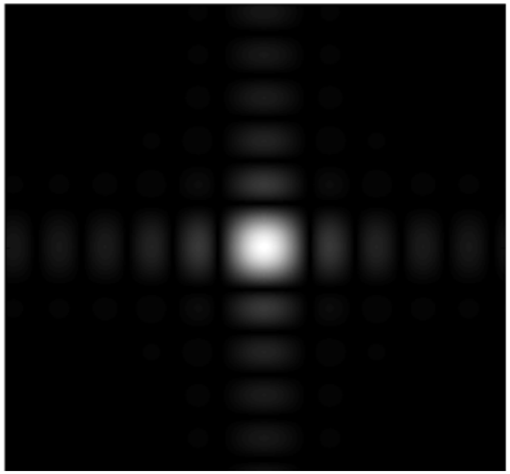
2-D Magnitude Plots



"Analytic Image"

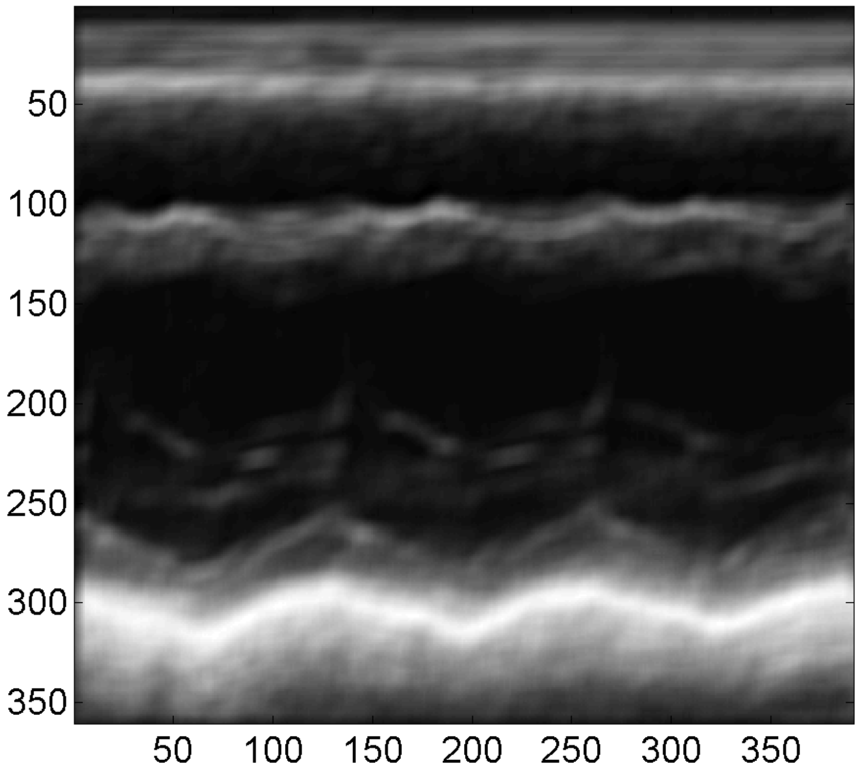


Filter 1

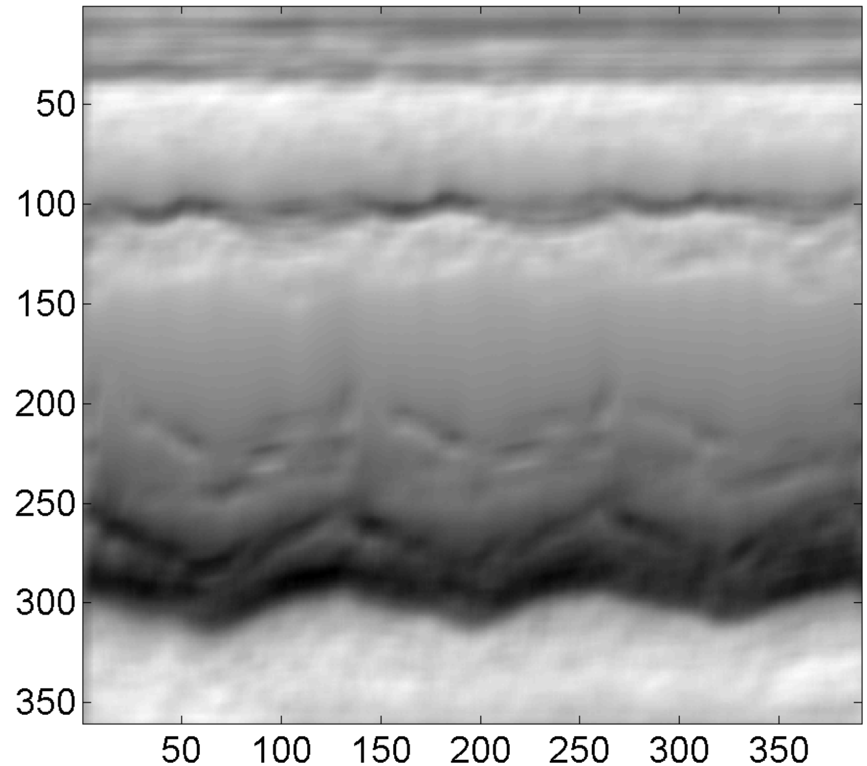


Filter 2

Filtered images through first filter

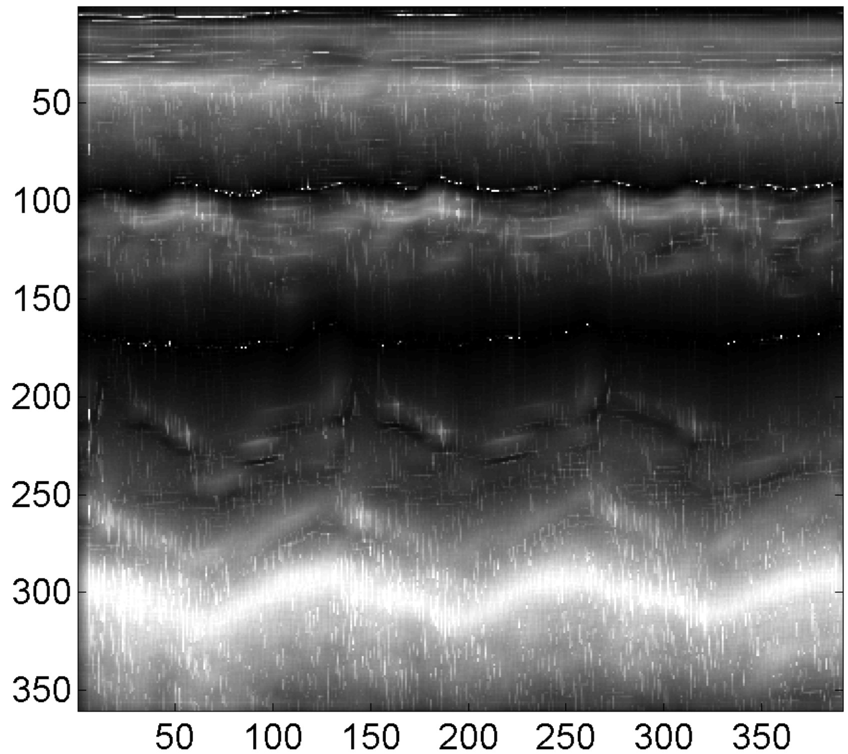


Negative real image

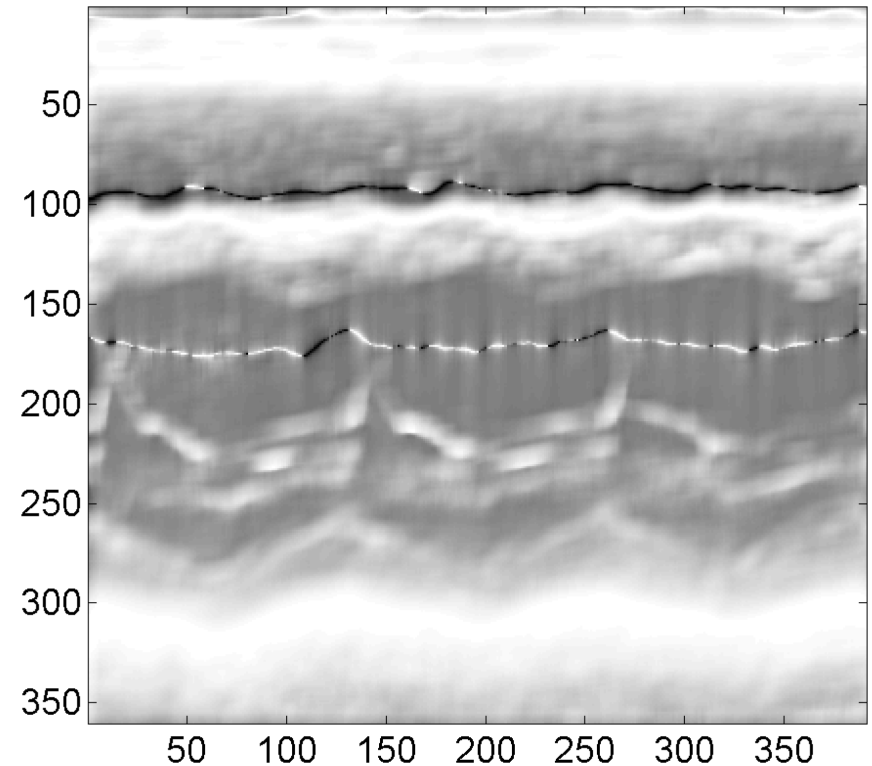


Negative imaginary image

AM-FM Estimates through first filter



Amplitude Image



FM image

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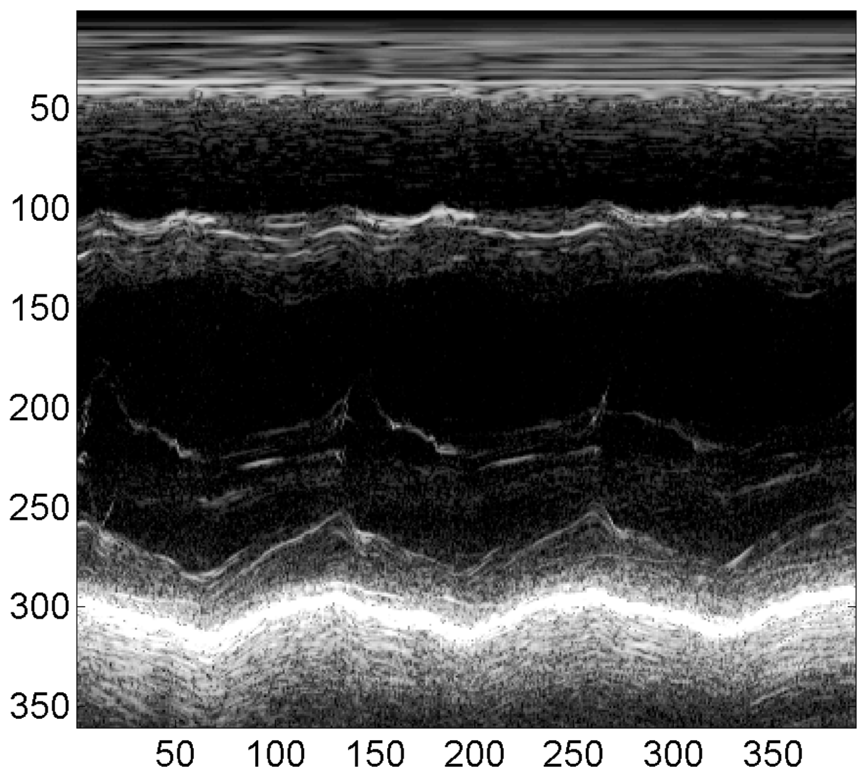
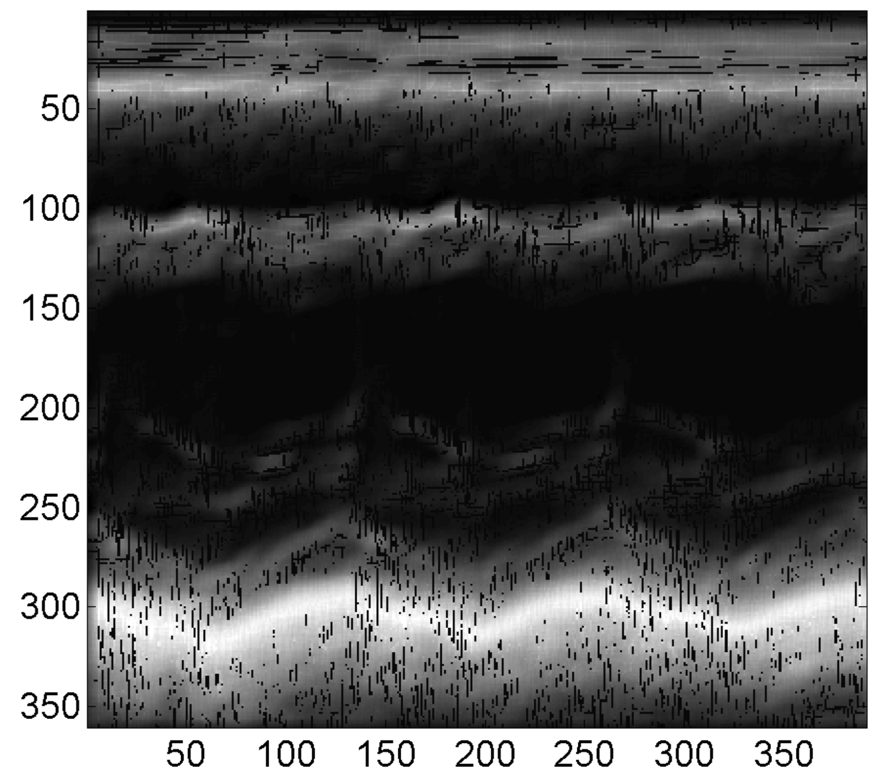
AM-FM Reconstruction over First Filter



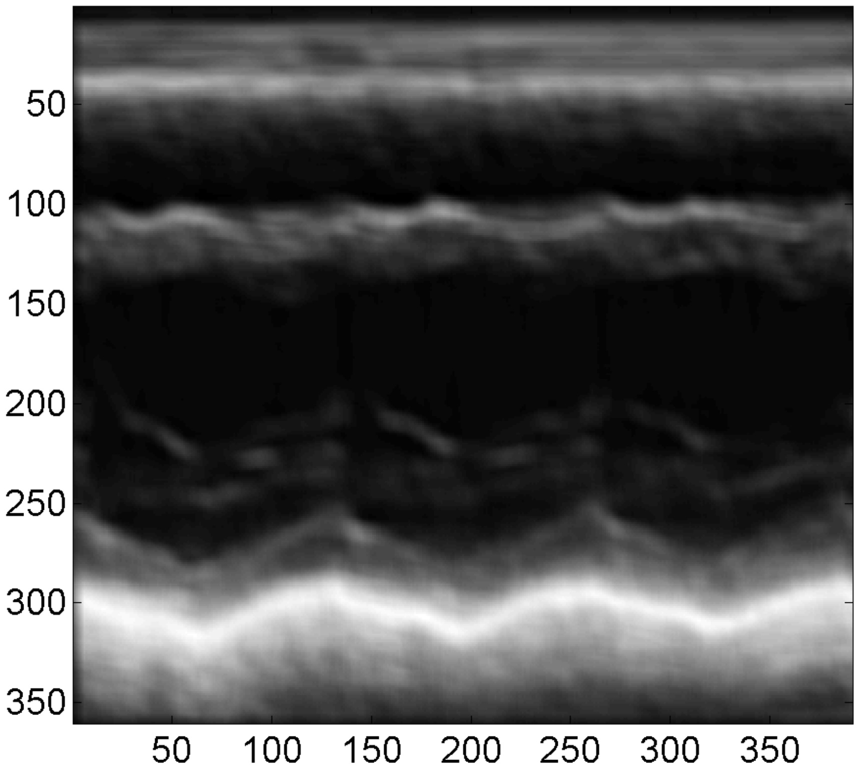
Original image



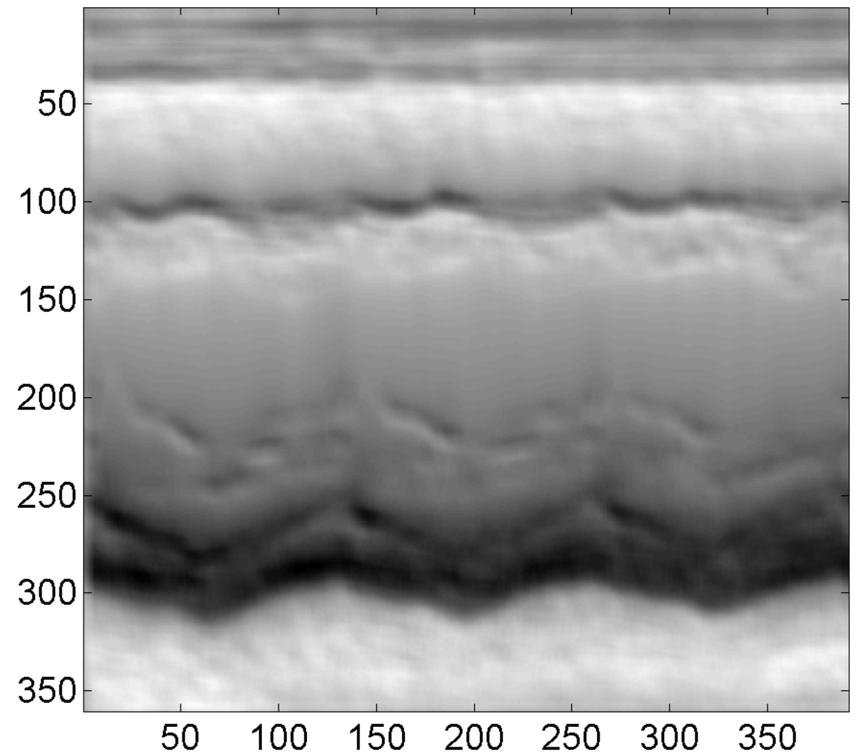
AM-FM reconstruction over filter 1.

Note black lines over regions where QEA “fails”.

Filtered image through second filter



Real Image



Imaginary Image

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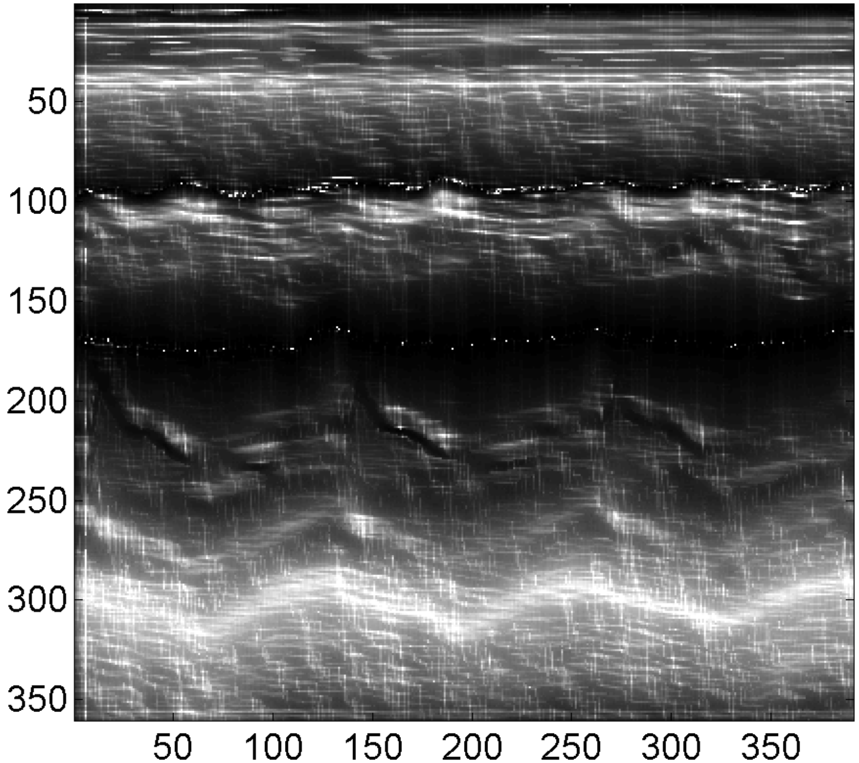
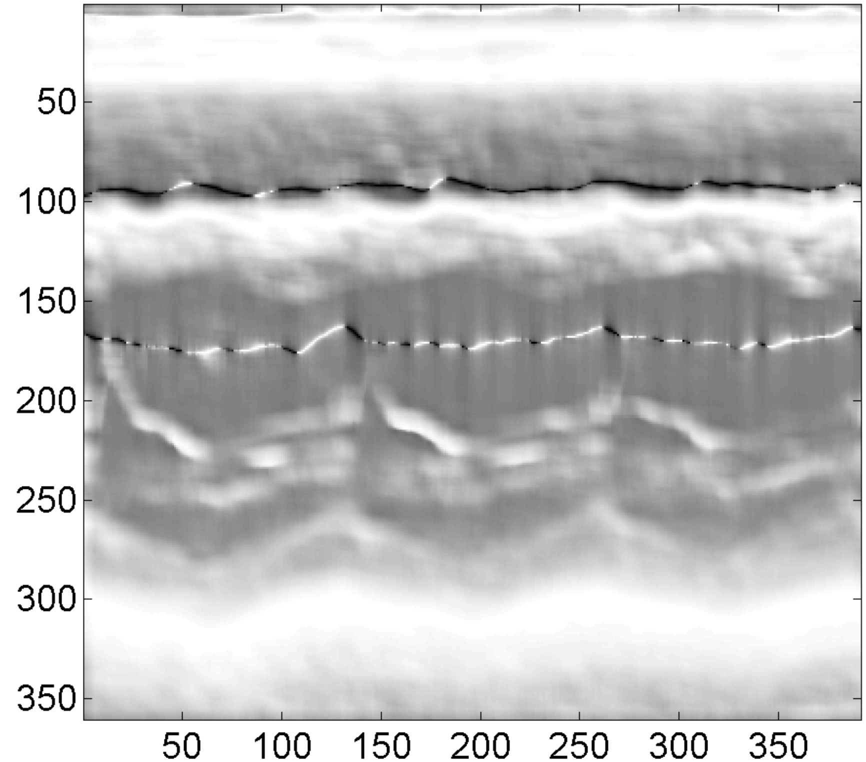
AM-FM Estimates through second filter

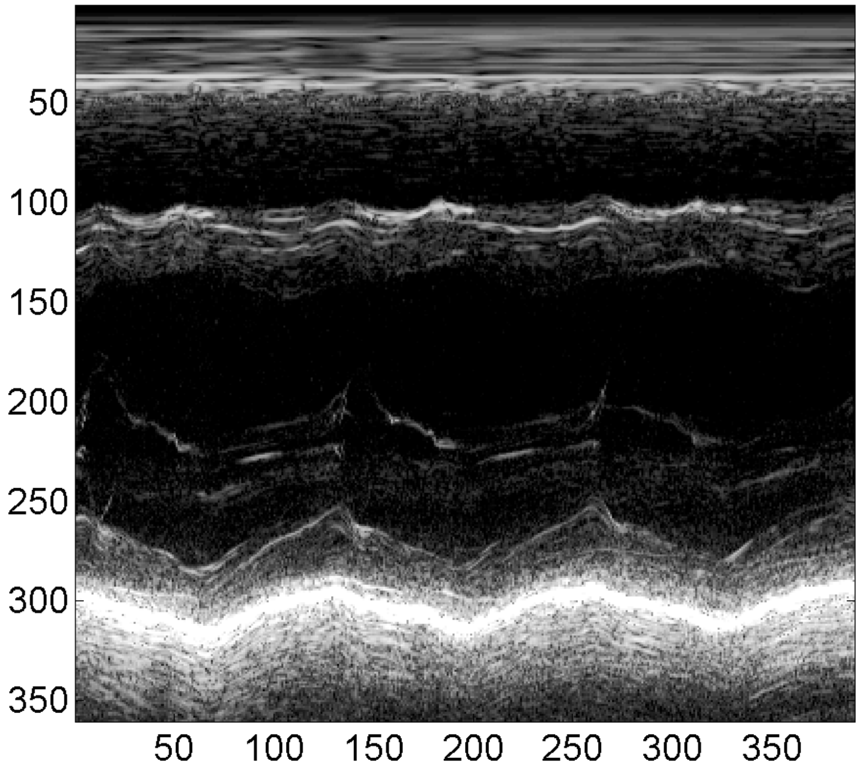


Amplitude Image

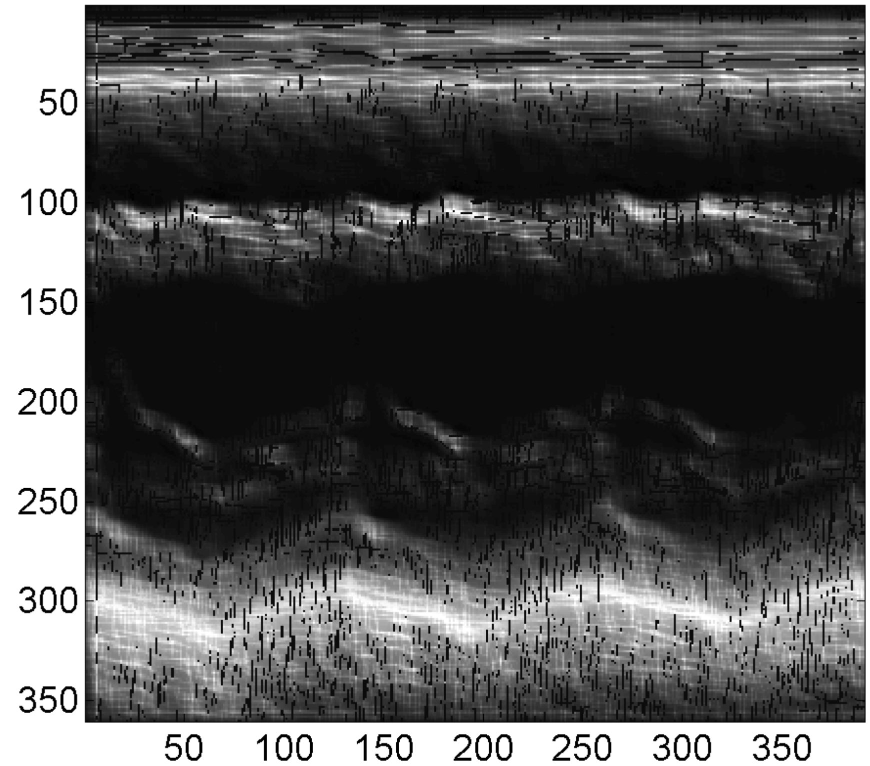


FM Image

AM-FM Reconstruction over Second Filter



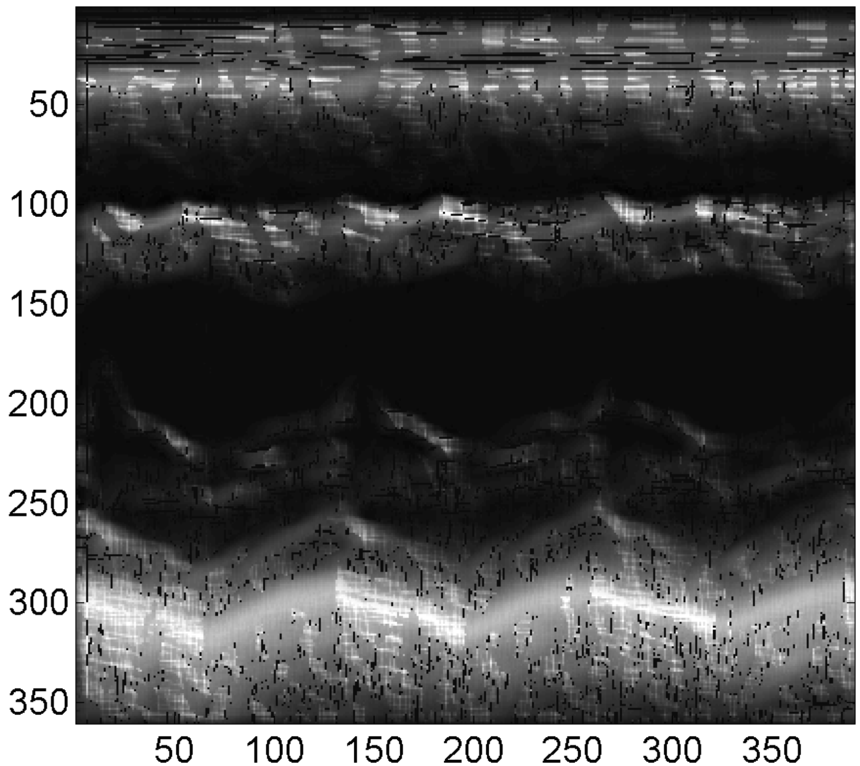
Original image

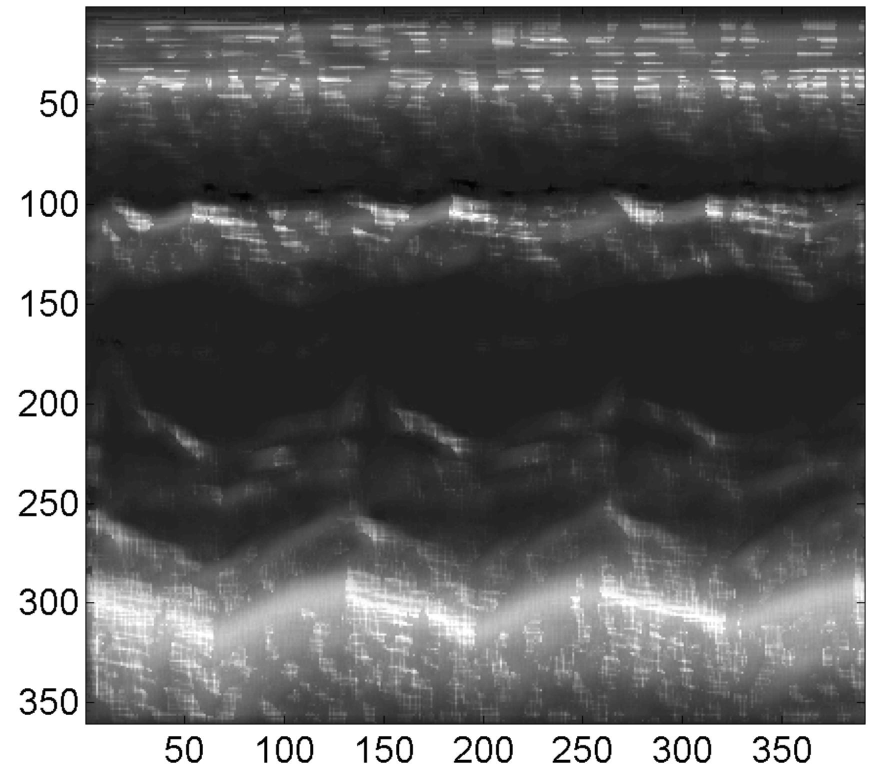


AM-FM Reconstruction over filter 2.

Note black lines over regions where "QEA" fails.

AM-FM Reconstructions over Both Filters





Reconstruction where QEA "holds"

Note the limited number of regions where QEA "fails".

Reconstruction with smoothing over regions where QEA "fails"

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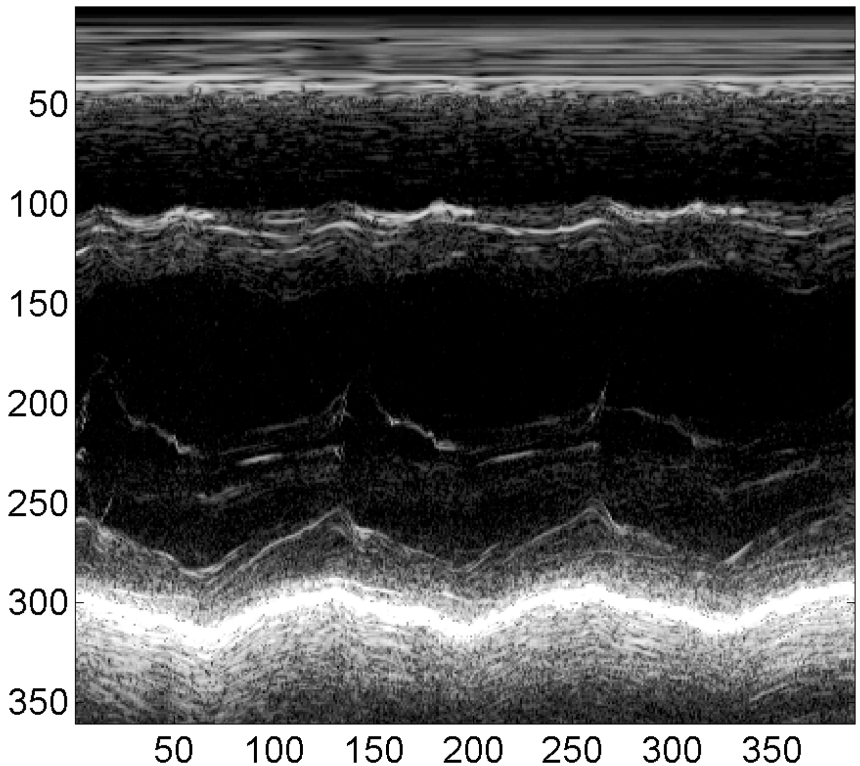
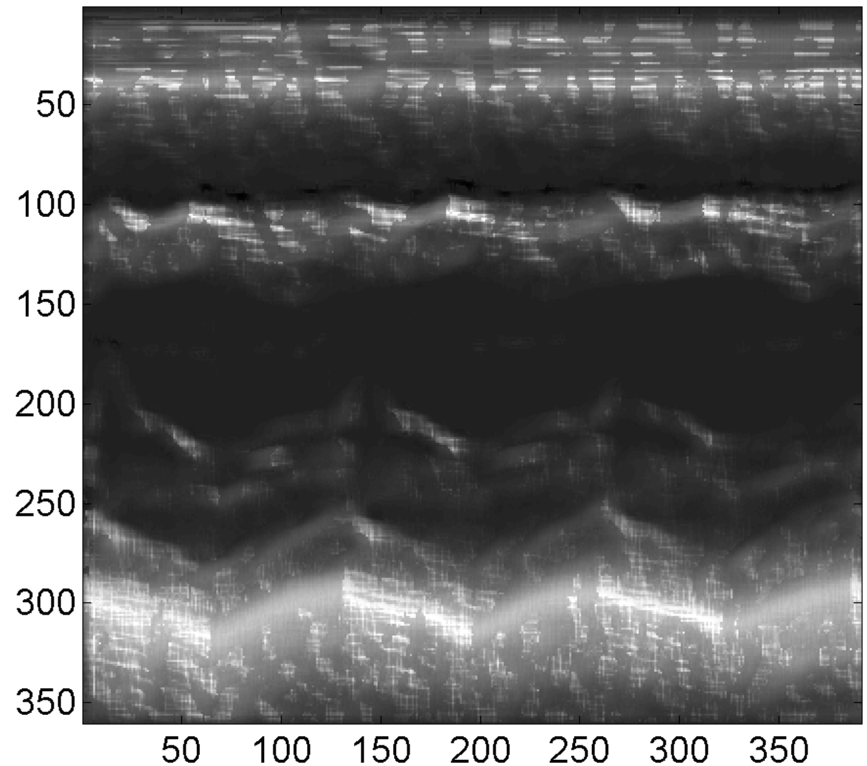
AM-FM Reconstruction vs Original Image

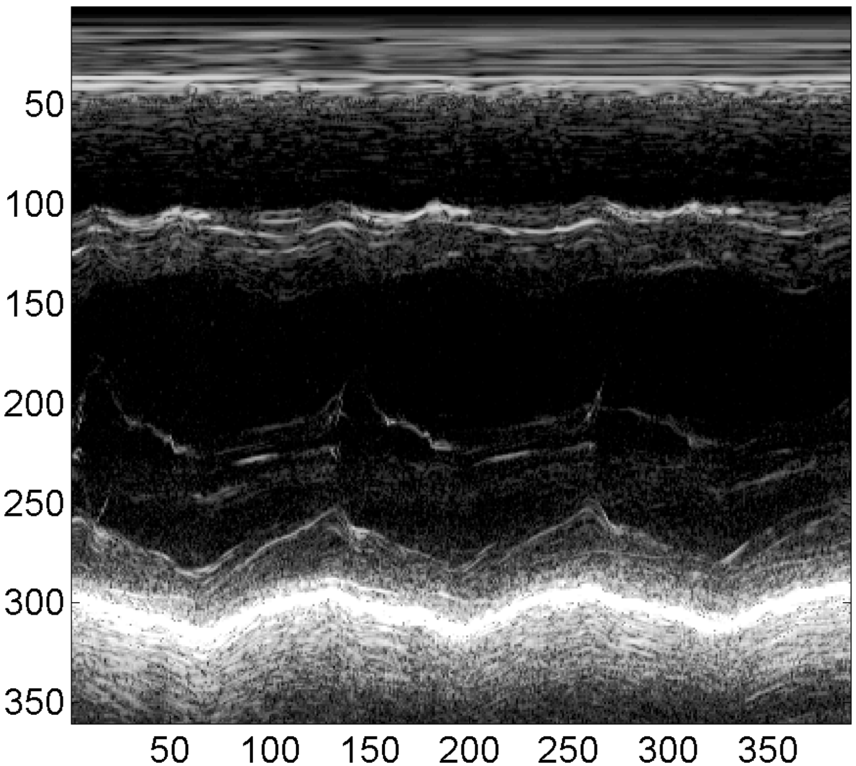


Original image

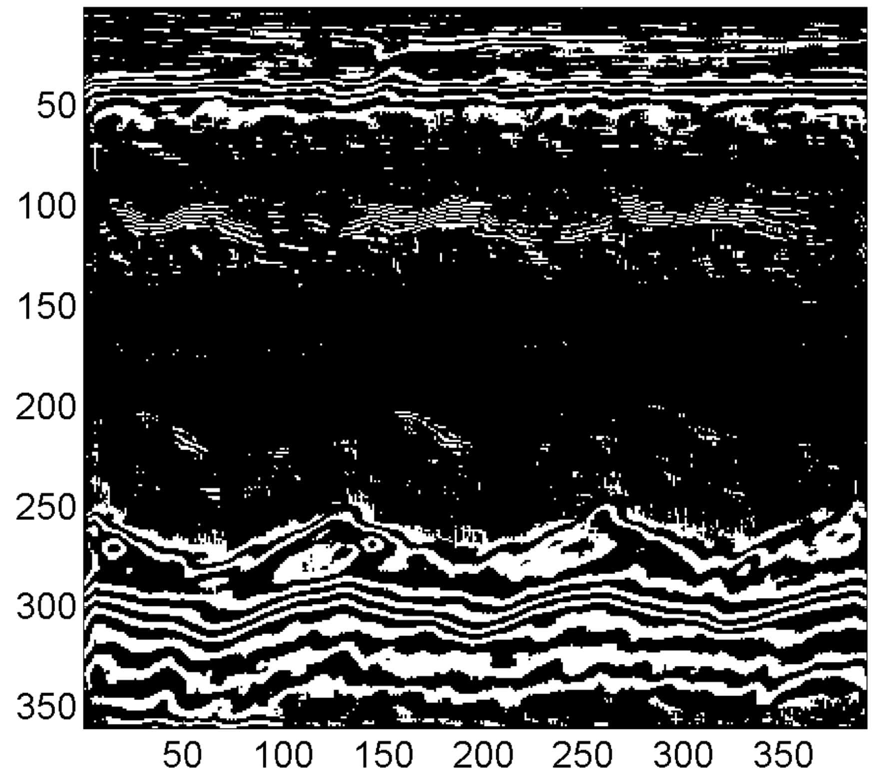


Reconstruction with smoothing over regions where QEA "fails"

AM-FM Reconstruction vs Harmonics Image (illustrating walls)



Original image



Sum of the first 20 FM harmonics thresholded and segmented for $a(x, y) > 100$ (first filter only)

Multidimensional Discrete-Space Orthogonal FM Transforms

Discrete-Space FM Transforms

Let g be a bounded M-dimensional signal defined on

$$Q = \{0, 1, \ldots, N-1\}^M.$$

We are interested in the *conditions* on the vector-valued phase-function $\mathbf{\Phi}$ so that g can be expressed as

$$g(\mathbf{n}) = N^{-M/2} \sum_{\mathbf{k} \in Q} g(\mathbf{k}) \exp\left[\, j \frac{2\pi}{N} \mathbf{k} \cdot \mathbf{\Phi}(\mathbf{n}) \,\right].$$

where M denotes the number of dimensions.

FM Transform Theorem

We write

$$g(\mathbf{n}) = N^{-M/2} \sum_{\mathbf{k} \in Q} G(\mathbf{k}) \exp\left[\; j \tfrac{2\pi}{N} \mathbf{k} \cdot \mathbf{\Phi}(\mathbf{n}) \;\right],$$

where the FM spectrum $G(\cdot)$ is given by

$$G(\mathbf{k}) = N^{-M/2} \sum_{\mathbf{n} \in Q} g(\mathbf{n}) \exp\left[\; -j \tfrac{2\pi}{N} \mathbf{k} \cdot \mathbf{\Phi}(\mathbf{n}) \;\right]$$

if and only if:

For $\mathbf{\Phi}(\cdot) = (\, \phi_1(\cdot), \phi_2(\cdot), \ldots, \phi_M(\cdot) \,)$,
$\forall \mathbf{n}, \mathbf{p}, \mathbf{n} \neq \mathbf{p}, \exists i$ such that $\phi_i(\mathbf{n}) - \phi_i(\mathbf{p}) \neq 0 \pmod N$.

The FM-transform condition is satisfied if $\mathbf{\Phi}$ is a permutation on Q.

Orthogonal FM Transform Relation to DFT

If $\Phi(\cdot)$ is a permutation of Q, then
the M-dimensional FM transform is equivalent to:

A permutation of signal samples,
followed by the M-dimensional
Discrete Fourier Transform (DFT).

We want to find permutations that permute any given signal to one with highly concentrated FM spectra.

Signals with Compact Spectra

A *proper unidirectional periodic signal* $x(\cdot)$ is any signal on Q that satisfies:

- $x(\mathbf{n})$ depends only on the first coordinate
  $x(\mathbf{n}) = x(n_1), \quad where \quad \mathbf{n} = (\, n_1, \ldots, n_M \,)$
- For some positive integer T dividing N, $x(\cdot)$ is T-periodic: $x(n_1) = x(n_1 + T)$.
- $x(0), x(1), \ldots, x(T-1)$ are distinct.

Matching Arbitrary Signals (including Signals with compact Spectra)

If the permutation $\mathbf{\Phi}$ of Q minimizes

$$\sum_{\mathbf{k} \in Q} \left(x(\mathbf{\Phi}(\mathbf{k})) - t(\mathbf{k}) \right)^2$$

then we say that $\mathbf{\Phi}$ *matches* x to t.

We match a signal x to a target signal t using:

If $\mathbf{\Phi}$ sorts x and $\mathbf{\Psi}$ sorts t, then $\mathbf{\Phi} \circ \mathbf{\Psi}^{-1}$ matches x to t.

For sorting multidimensional signals, we first order the samples into a one-dimensional signal.

JPEG Modified for Implementing FM Transform

Encoder

Step 1. Compute Optimal Permutation
Step 2. Permute Image Samples
Step 3. Compress Optimal Permutation
Step 4. Apply DCT

Decoder

Step 1. Apply inverse DCT
Step 2. Uncompress Optimal Permuation
Step 3. Depermute image samples

One-bit Permutations for DCT Target

Sort the signal $y(\cdot)$ into $x(\cdot)$:

$$\overbrace{x_1 \leq \cdots \leq x_{N^2/2}}^{\text{treated as } v_2} \leq \overbrace{x_{N^2/2+1} \leq \cdots \leq x_{N^2}}^{\text{treated as } v_1}$$

Need only store a single bit per signal sample in $y(\cdot)$ (meaning v_1 or v_2).

Then, map the first v_1 value to the first location in the matrix below, and so on for v_2 and the rest of the values:

$$x_P = \begin{bmatrix} v_1 & v_2 & v_2 & v_1 & \dots & v_1 & v_2 & v_2 & v_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_1 & v_2 & v_2 & v_1 & \dots & v_1 & v_2 & v_2 & v_1 \end{bmatrix}$$

Two-bit Permutations for DCT Target

Sort the signal $y(\cdot)$ into $x(\cdot)$:

$$\overbrace{x_1 \leq \cdots \leq x_{N^2/4}}^{\text{treated as } v_4} \leq \overbrace{x_{N^2/4+1} \leq \cdots \leq x_{N^2/2}}^{\text{treated as } v_3}$$

$$\leq \cdots \leq \overbrace{x_{3N^2/4+1} \leq \cdots \leq x_{N^2}}^{\text{treated as } v_1}$$

Need only store two bits per signal sample in $y(\cdot)$ (meaning v_1, v_2, v_3, or v_4).

Then, permute the signal samples to match the pattern

$$x_P = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_4 & v_3 & v_2 & v_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_1 & v_2 & v_3 & v_4 & v_4 & v_3 & v_2 & v_1 & \cdots \end{bmatrix}$$

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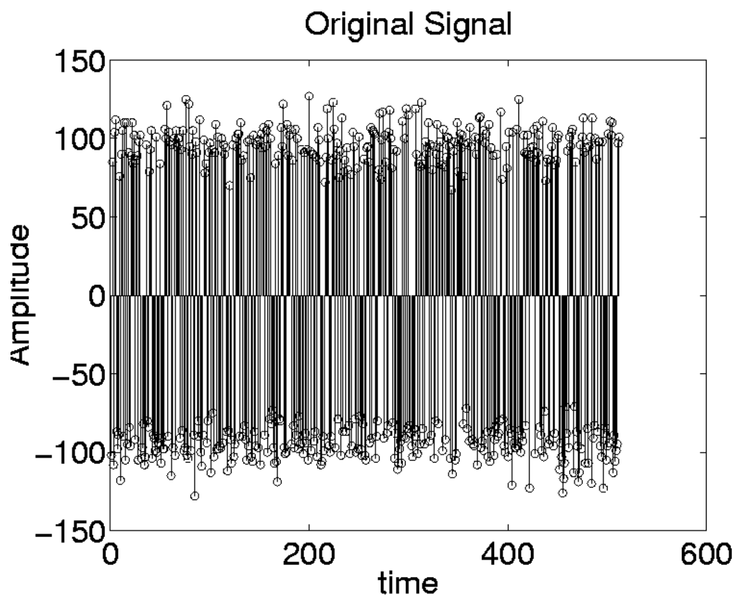
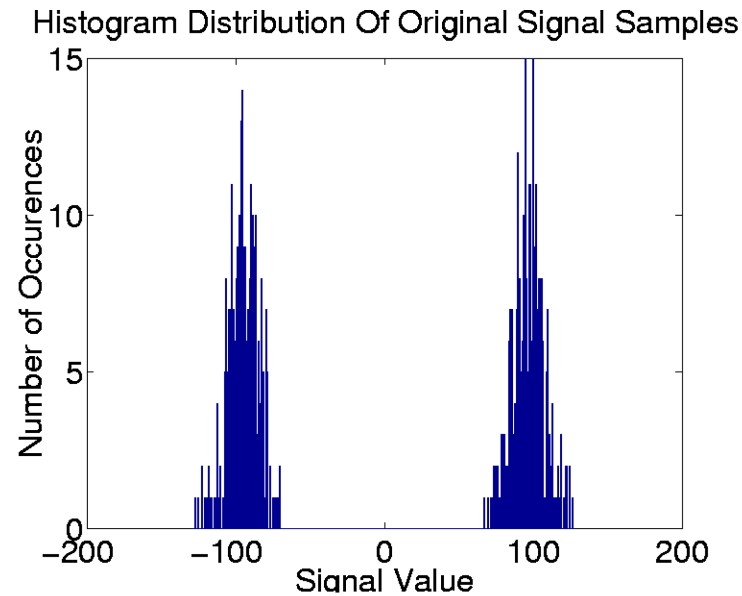
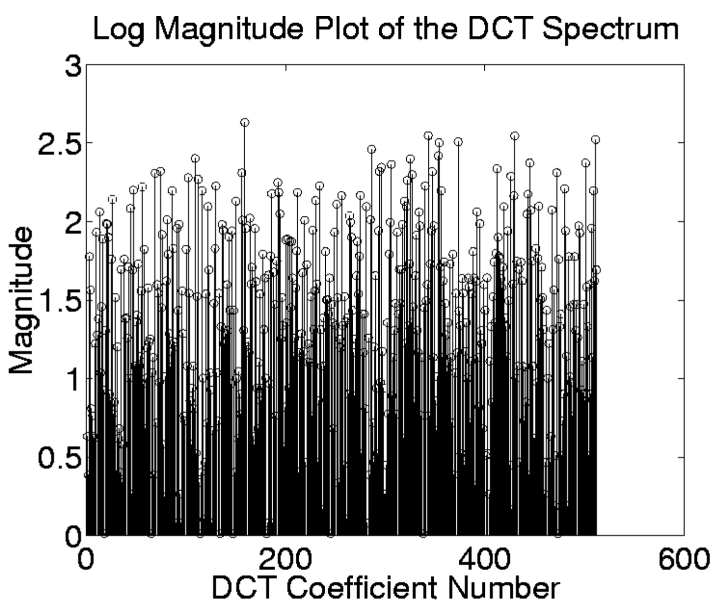
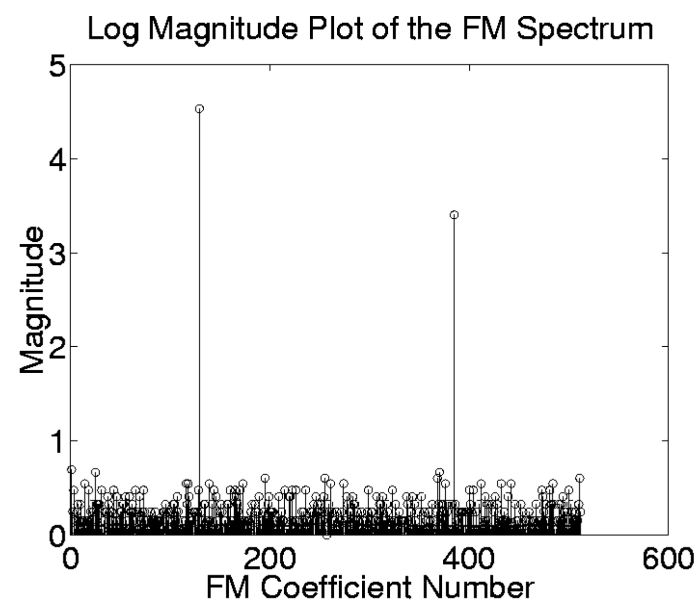
Spectral Energy Compaction

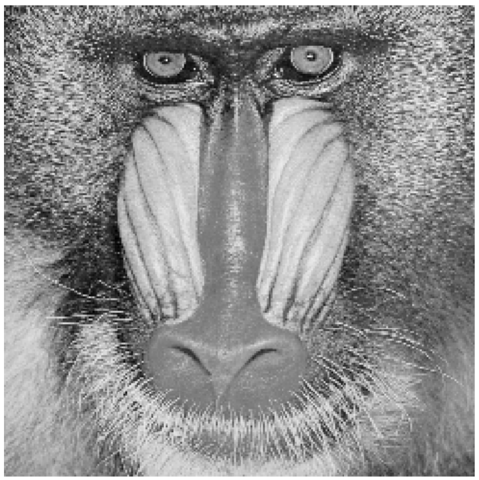


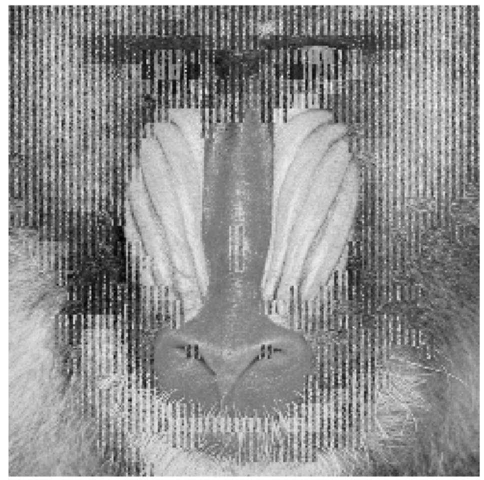


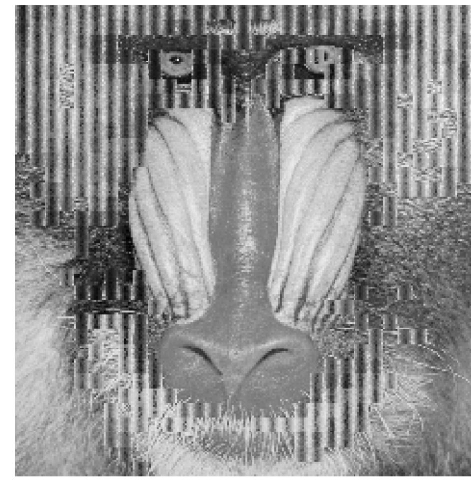


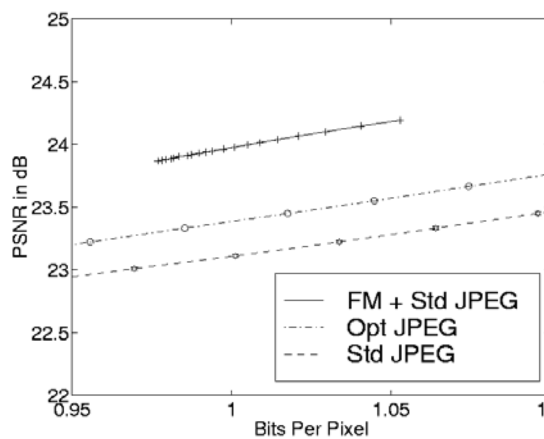


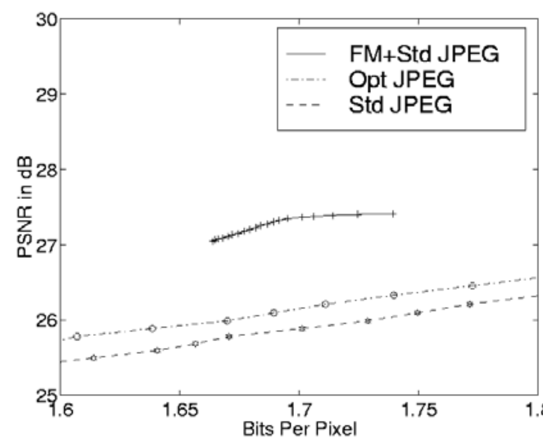
Orthogonal FM Transform: Some Image Compression Results











Future work in Multispectral Image Compression?

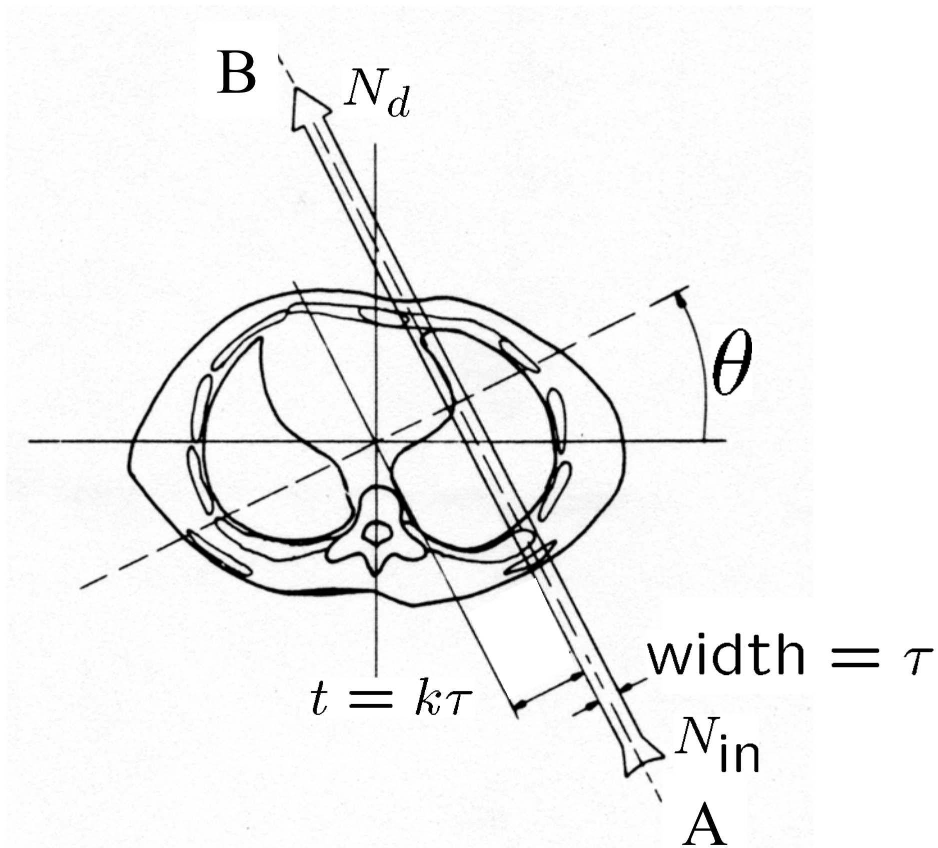
Hypothesis

If the pixels represent the same structure, then they require the same permutation irrespective of the spectral band.

Hence, high compression should be possible by Effectively encoding permutation changes between bands.

Application to Multidimensional Discrete Fourier Transform and Computerized Tomography

The Transmission Model



$$P_\theta(\kappa\tau) = \int_{\text{ray path AB}} \mu(x, y)dx = \ln \frac{N_{in}}{N_d}$$

From “Principles of Computerized Tomographic Imaging” by Kak and Slaney.

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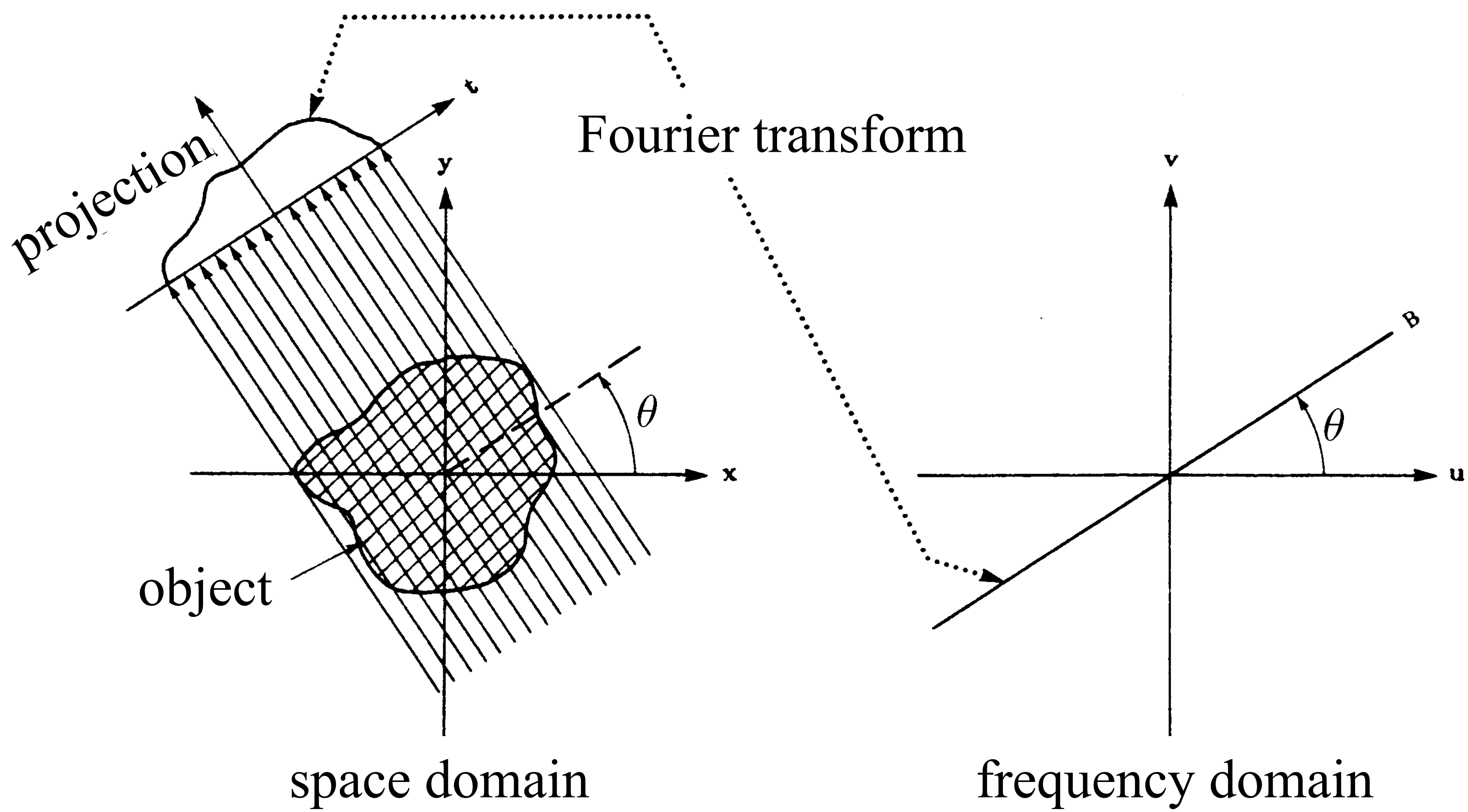
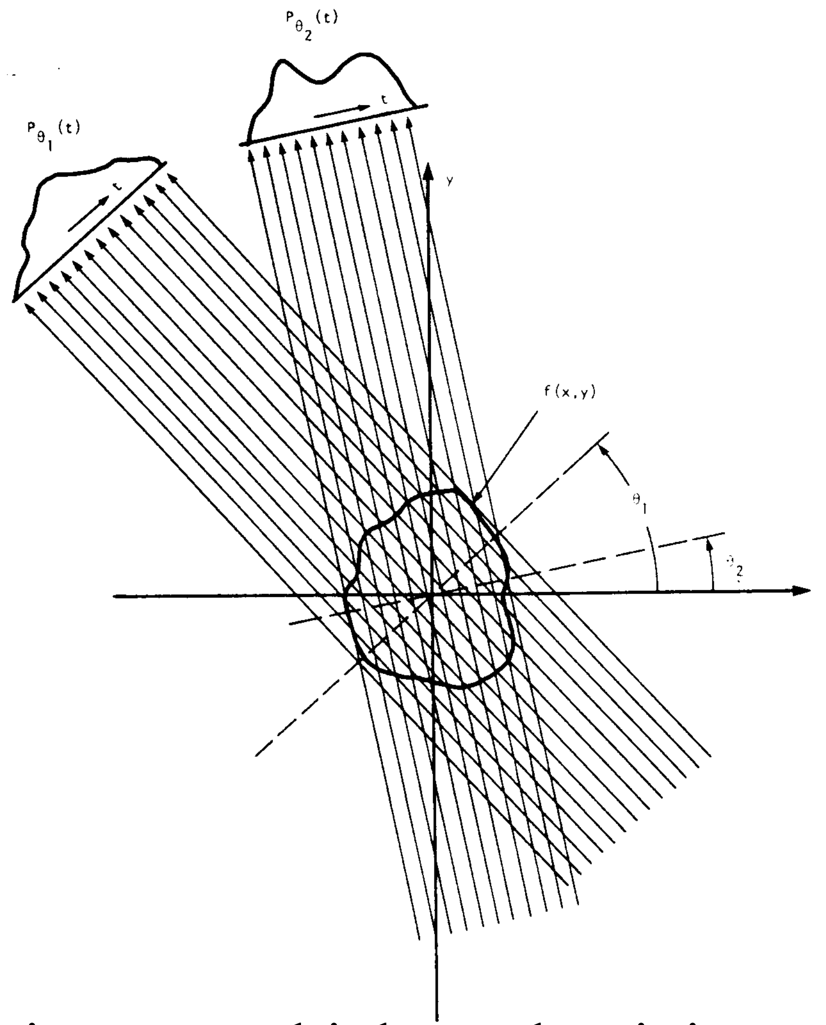
The Fourier Slice Theorem



Each Projection can be used to compute a line spectrum in the frequency domain.

From "Principles of Computerized Tomographic Imaging" by Kak and Slaney.

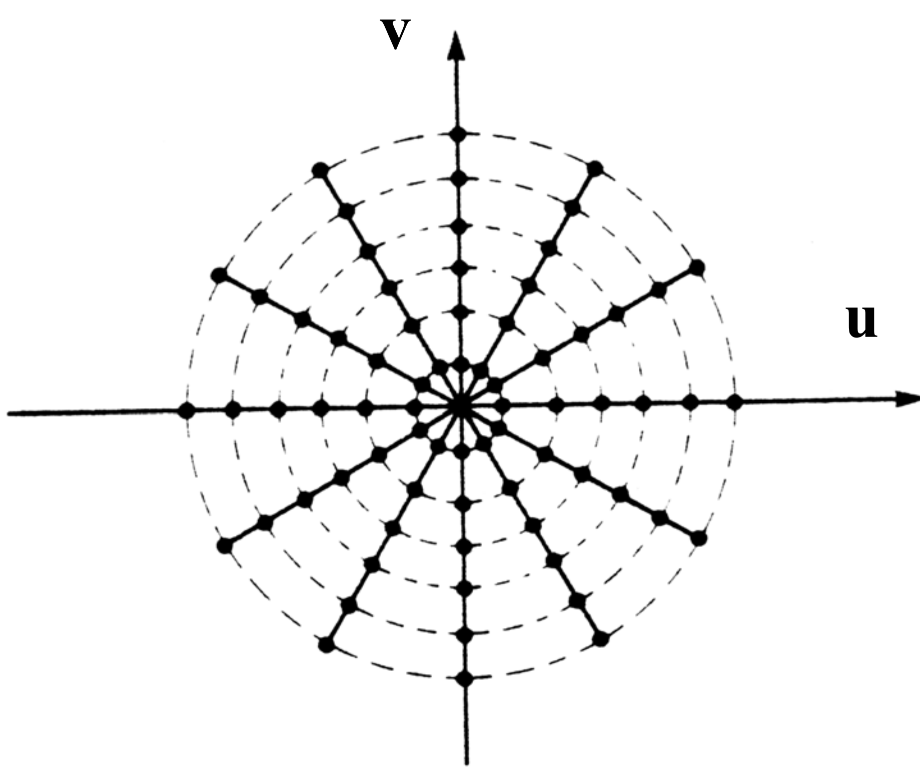
Parallel Projections



By taking projections at multiple angles, it is possible to reconstruct an image, and invert the projection operation.

From “Principles of Computerized Tomographic Imaging” by Kak and Slaney.

Frequency-Domain Sampling

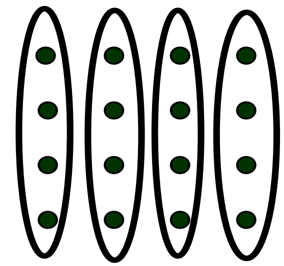


To reconstruct the image, interpolate to estimate the 2-d spectrum, and take the inverse 2-d FFT.

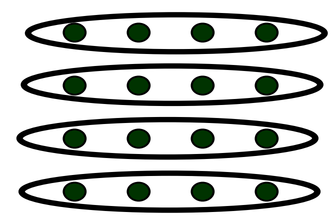
From "Principles of Computerized Tomographic Imaging" by Kak and Slaney.

Three Directions of Interest

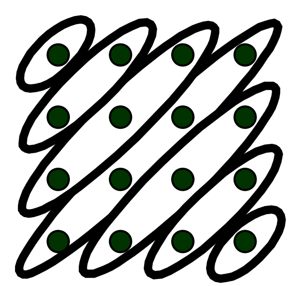
For Horizontal Frequencies:



For Vertical Frequencies:



For Diagonal Frequencies:



Covering the DFT Spectrum

Definition. A set S' is said to cover S if for every $\mathbf{v} \in S$, we can find a $\mathbf{v}' \in S'$, and an integer $a > 0$, such that $\mathbf{v} = a\mathbf{v}'$.

Define V by $V = \{\mathbf{v} \in S | \exists i \quad \text{such that} \quad v_i = 1\}$.

Theorem. Let $N_i = 2^{p_i}$, where $p_i > 0$. Then V covers S.

Two-Dimensional DFT Computation

Cover S **(in two dimensions) using:**

$$V_1 = \{(i,1) \mid i = 0, 1, 2, \ldots, N_1 - 1\}$$

$$V_2 = \{(1,i) \mid i = 0, 2, 4, \ldots, N_2 - 2\}$$

$$V_C = V_1 \cup V_2$$

Only $\frac{3N}{2} = N + \frac{N}{2}$ ***1-D FFTs (for square images).***

Three-Dimensional DFT Computation

Cover S (in three dimensions) using:

$$V_1 = \{(1, i, j) \mid 0 \le i \le N_2 - 1,\ 0 \le j \le N_3 - 1\}$$

$$V_2 = \{(j, i, 1) \mid j = 0, 2, 4, \ldots, N_1 - 1, 0 \le i \le N_2 - 1\}$$

$$V_3 = \{(i, j, 1) \mid i = 0, 2, 4, \ldots, N_1 - 1,\ j = 0, 2, 4, \ldots, N_2 - 1\}$$

$$V_C = V_1 \cup V_2 \cup V_3$$

Only $\frac{7N^2}{4} = N^2 + \frac{N^2}{2} + \frac{N^2}{4}$ ***1-D FFTS (for*** $N \times N \times N$ ***images).***

Minimum Cardinality Result I

Theorem 2. The cardinality of V_C, is minimal in the sense that:

There does not exist $V_C^{'}$ that covers S, yet satisfying $\left|V_C\right| > \left|V_C^{'}\right|$.

<u>The 2-D and 3-D Directional Decompositions Exhibit the Minimum number of 1-D FFTs.</u>

Minimum Cardinality Result II

- Each FFT can be modified to compute the spectrum along spectral frequencies that were not computed before.

- Minimal computation can be achieved by requiring that the number of computed DFT frequencies is equal to the number of possible DFT frequencies in the image.

Eg: We use $3N/2$ "Decimated in Frequency" FFTs to compute a total of N^2 discrete frequencies.

Concluding Remarks

- AM-FM analysis holds great promise for **continuous-scale analysis problems** (eg: M-mode ultrasound, …)

- Multidimensional DFT work will be applied in image restoration, **fast**, but also:

  Numerical accuracy:

  uses a single FFT per discrete frequency, irrespective of dimension!

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